

From string theory to large N QCD

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August, 2010.

A thesis submitted to McGill University
in partial fulfillment of the requirements of the degree of Doctor of Philosophy

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Abstract

We propose the dual gravity of a non conformal gauge theory which has logarithmic running of couplings in the IR but becomes almost conformal in the far UV. The theory has matter in fundamental representation, non-zero temperature and under a cascade of Seiberg dualities, can be described in terms of gauge groups of lower and lower rank. We outline the procedure of holographic renormalization and propose a mechanism to UV complete the gauge theory by modifying the dual geometry at large radial distances. As an example, we construct the brane configuration and sources required to attach a Klebanov-Witten type geometry at large r to a Klebanov-Strassler type geometry at small r . Using the supergravity description for the dual geometry, we compute thermal mass of a fundamental ‘quark’ in our theory along with drag and diffusion coefficients of the gauge theory plasma. We compute the stress tensor of the gauge theory and formulate the wake a probe leaves behind as it traverses the medium. Transport coefficient shear viscosity η and its ratio to entropy η/s are calculated and finally we show how confinement of ‘quarks’ at large separation can occur at low temperatures. We classify the most general dual geometry that gives rise to linear confinement at low temperatures and show how quarkonium states can melt at high temperatures to liberate ‘quarks’.

Résumé

Nous proposons une théorie de la gravité qui correspond à une théorie de jauge avec des constantes de couplage à comportement logarithmique dans l'infrarouge et devenant quasi-conforme dans l'ultraviolet profond. La théorie contient la matière dans la représentation fondamentale, a une température non-nulle et peut-être décrite en terme de groupes de jauge de rangs de plus en plus petits par une cascade de dualités de Seiberg. Nous discutons brièvement du processus de renormalisation holographique et proposons un mécanisme pour compléter la théorie de jauge dans l'ultraviolet. Comme exemple, nous construisons une configuration de membranes et de sources qui joignent une géométrie de Klebanov-Witten à grand r à une géométrie de Klebanov-Strassler à petit r . En utilisant la supergravité pour décrire la géométrie duale, nous calculons la masse thermique d'un quark fondamental ainsi que les coefficients de traînée et de diffusion du plasma de la théorie de jauge. Nous évaluons le tenseur de stress de la théorie de jauge et quantifions le sillon laissé par une sonde traversant le milieu. Nous calculons le rapport de la viscosité de cisaillement à la densité d'entropie, η/s . Finalement, nous montrons comment la géométrie duale la plus générale possible peut donner lieu à la fois au confinement linéaire de paires quark-antiquark à basse température et à la dissolution du quarkonium à haute température.

Dedication

To my brother Mashrur Mia.

Acknowledgements

First and foremost, I would like to thank my advisor Professor Charles Gale for believing in me and being there when it mattered the most. I would not be a graduate student if it wasn't for Charles. The research for my doctoral studies would not have been possible without the guidance of Professor Keshav Dasgupta. The effort and hours he spent on a daily basis explaining the intricacies of string theory to someone like me with the minimal knowledge and that too for two and a half years, is truly inspiring. I cannot imagine the completion of this work without Keshav. I am in debt to Professor Sangyong Jeon, for the training he has given me in the last six years, to question, to be critical and most importantly, to be more of a physicist.

I would not be a scientist, if it wasn't for my mom and dad, who brought me to this world and taught me how to think and raised me to who I am. Everything that is good and creative in me is from them. I have to thank my brother Sa'ad for being my moral guru, the guardian he has always been and for enduring all my irregularities, making our apartment our home. I'm truly blessed to have my better half Roshni to be there always, I would be totally lost without her. I have to thank my office mates Rhiannon, Aaron, Paul and Simon, my colleagues Alisha, Qin, Nima and Hoi for all the discussions on physics and non-physics topics that keep us going. Thanks to Andrew Frey and Alejandra for always answering my questions with the best insight into the subject.

My cousins and family in Montreal, Choyon, Ayon, Ruben bhaiya, Juthi bhaabi, Evan, Salek bhai, Sani- was always there to make Montreal my home. Thanks Yuri for lending your ears, painting my face and kicking me out to freeze. I'm grateful to all my brothers and sisters in McGill- Sayem, Takshed, Khalid, Saquib bhai, Rafa Apu, Sadaf Apu, Rafi bhai, Ruhan bhai and Mehdi bhai- you have always managed to entertain and put a smile on my face.

Finally, I thank Allah, the almighty, for everything.

Statement of originality

The thesis is based on my research done in collaboration with Keshav Dasgupta, Charles Gale and Sangyong Jeon and the results presented constitute original work that appeared in the following articles:

- **M. Mia**, K. Dasgupta, C. Gale and S. Jeon, “*Five Easy Pieces: The Dynamics of Quarks in Strongly Coupled Plasmas*”, Nucl. Phys. B **839**, 187 (2010), 107pp, [arXiv:0902.1540 [hep-th]].
- **M. Mia**, K. Dasgupta, C. Gale and S. Jeon, “*Toward Large N Thermal QCD from Dual Gravity: The Heavy Quarkonium Potential*”, Phys. Rev. D **82**, 026004 (2010), 28pp, [arXiv:1004.0387 [hep-th]].
- **M. Mia**, K. Dasgupta, C. Gale and S. Jeon, “*Heavy Quarkonium Melting in Large N Thermal QCD*”, in press for Phys. Lett. B (2010), doi:10.1016/j.physletb.2003.10.071, 15pp, arXiv:1006.0055 [hep-th].
- **M. Mia** and C. Gale, “*Jet quenching and the gravity dual*”, Nucl. Phys. A **830**, 303C (2009), 4pp, [arXiv:0907.4699 [hep-ph]].
- **M. Mia**, K. Dasgupta, C. Gale and S. Jeon, “*Quark dynamics, thermal QCD, and the gravity dual*”, Nucl. Phys. A **820**, 107C (2009), 4pp.

All the original calculations presented here were performed by me except the ones concerning arbitrary UV completions, the explicit form of the fluxes and the proof of melting of the potential, which were done by my collaborators.

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Chapter 1

Introduction

Since the discovery of a unified description of strong, weak and electromagnetic interaction back in the 1970's, much activity was centered around its consequences. The foundation of the unification lies in the principle of local gauge symmetry where interactions between particles are mediated by gauge fields. Through the mechanism of spontaneous symmetry breaking as in the Higgs model [1]-[4], Weinberg [5] and Salam [6] showed that at low energies only photons and neutrinos remain massless while vector bosons responsible for weak interaction acquire mass. In the far UV the vector bosons become massless and strong and weak interactions become long range- just like electromagnetic interaction. Thus at high energies symmetry between strong, weak and electromagnetic force gets restored [6][5].

On the other hand phase transition is a process which changes the symmetry of the system and before or after the phase transition, symmetry is broken [7]. This suggests the existence of various phases in the theory of elementary particles. In particular, highly dense matter under extreme temperature and pressure should exhibit various phases as temperature and density is altered. By heating nuclear matter up to extreme temperature and pressure, one expects to create a new phase of matter, a plasma of quarks and gluons where quarks become massless with strong, weak and electromagnetic interaction becoming long range, restoring the symmetry between them. The estimates for the critical temperature T_c for phase transition in renormalizable gauge theories were first made by [8]-[10] and there is an extensive literature

on the phase structure of quark matter along with studies of critical temperature [11]-[14].

These theoretical expectations for a new phase of matter known as the Quark Gluon Plasma (QGP) led to the Relativistic Heavy Ion Collider (RHIC) program where heavy nuclei are made to collide at relativistic velocities. The matter formed in the early stages of the collision has indeed been identified not as a collection of color neutral hadrons, but a new state of dense matter [15]-[18]. The relativistic fluid created at RHIC cannot be described in terms of hadronic degrees of freedom and one may conclude that the energies reached by the experiments give rise to a plasma of quarks and gluons i.e. QGP. But what are the properties of this fluid? In particular, is the coupling between the constituents of the fluid strong or weak? The answer to this is crucial as it will determine the theoretical tools to study the fluid. At weak coupling, perturbative methods can be useful whereas at strong coupling, effective or lattice field theory need to be applied.

To address this issue, first observe that one of the key characteristics of a fluid is how it responds to pressure variation and flows to equilibrium state. At the heavy ion collider, the overlapping region between two heavy nuclei is of elliptic shape with one axis longer than the other- thus creating anisotropic pressure. The fluid will of course try to equilibrate to spherical symmetric configuration and this leads to elliptic flow. This flow as observed in the experiments at RHIC [19]-[22] is well described by ideal hydrodynamics which assumes that the QGP fluid is strongly coupled. This means perturbative field theory techniques are not useful to analyze this strongly coupled system and one usually studies the theory on the lattice or using effective field theory. While the former becomes quite challenging with lattice simulations limited by computational ability of the numerical method, the latter relies on effective Lagrangians which are only approximations and could lead to incomplete results.

But these are not the only tools to describe a strongly coupled field theory. One may apply the principle of holography which relates the Hilbert space of a gauge theory with that of a theory of gravity, to study strongly coupled field theories. The key observation was made by Maldacena in the late 1990's while studying anti de

Sitter black hole solutions and how they arise from D brane configurations [23]. He conjectured that strongly coupled $\mathcal{N} = 4$ supersymmetric conformal field theory is dual to a theory of weakly coupled gravitons describing $AdS_5 \times S_5$ geometry, where AdS_5 is the five dimensional anti de Sitter space and S_5 is the five sphere. This duality known as the AdS/CFT correspondence opened a whole new avenue to study strongly coupled quantum field theories using weakly coupled classical gravity and is only a part of a more general correspondence between gauge theory and geometry.

In this thesis we extend the AdS/CFT correspondence to incorporate renormalization group flow in the gauge theory and study the system at strong coupling using weakly coupled dual gravity. We propose the dual geometry for a strongly coupled gauge theory which has logarithmic running of couplings in the far IR but becomes almost conformal in the far UV. The gauge theory has matter in fundamental representation and has non-zero temperature- which is achieved by introducing a black hole in the dual geometry. Using the weakly coupled gravity description for the strongly coupled field theory, one can easily compute expectation values of various gauge theory operators that are extremely difficult to obtain using conventional techniques. By computing quantum correlation functions using dual classical action, one can analyze the thermodynamic properties and learn about the kinematics of strongly coupled gauge theory plasma. But how does the rich structure of quantum field theory get exact description in terms of a simple classical theory of gravity? To be more precise, what are the limits of validity of the correspondence?

It turns out that for a field theory at strong coupling to have a description in terms of weakly coupled gravity, the number of colors N in the gauge theory must be very large. In the large N limit, it seems that the field theory has a classical description in terms of dual gravity where quantum correlation functions can be exactly calculated. At first glance this simplification might appear to be quite ‘accidental’ but a careful analysis can shed light on the issue. The answer may be linked to an observation made by ’t Hooft who suggested that the number of colors N in a gauge theory can be thought of as an expansion parameter [24]. There are planar and non planar diagrams that contribute to the propagator and in the limit $N \rightarrow \infty$, only

planar diagrams survive. This is because non planar diagrams are suppressed by $\mathcal{O}(1/N^k)$, $k \geq 1$, relative to planar ones [24][25], resulting in a drastic simplification of the propagator. While at finite N propagators and subsequently S matrix elements are extremely difficult to compute due to the large number of non planar diagrams at various loop order, in the large N limit, the theory has a rather simple description. This simplification from a field theory analysis is completely consistent with our dual gravity description where theory can be described by classical gravity.

The thesis is organized as follows: In chapter two, after a brief discussion on AdS/CFT correspondence, we describe in some detail gauge theories that arise from various brane configurations and then present their dual geometries. We outline the procedure of holographic renormalization of non conformal gauge theories which have dual gravity description and then discuss how to UV complete the theory by attaching geometries. In the final section of chapter two, we give a specific example of such UV completion by considering localized sources which allow one to attach asymptotic AdS geometry at large r , to a Klebanov-Strassler type geometry at small r . Using the dual geometries of chapter two, in chapter three we compute various gauge theory quantities crucial in analyzing a plasma. We compute the thermal mass of ‘quark’ in fundamental representation, the drag it experiences as it moves through the plasma and the wake it leaves behind as it traverses the medium. We also compute the momentum broadening of a fast moving jet along with transport coefficients such as shear viscosity and its ratio to entropy. Finally we show how confinement can be achieved using dual gravity and propose the most general dual geometry that can realize linear confinement at large distances. We conclude the thesis with a brief summary of our results along with future directions that can be explored.

Chapter 2

The Gauge/Gravity Correspondence

Quantum field theory with gauge symmetry has been extremely successful in describing the dynamics and collective excitations of highly energetic particles at very short distance scales. On the other hand, Einstein's theory of gravity has been crucial in understanding the dynamics of massive objects separated by large distances under the force of gravity. But what is the connection between these two seemingly distinct theories at opposite distance scales? General relativity dictates that any stress energy will couple to gravitons and thus given a field theory one can always compute the gravitons sourced by the stress tensor of the field theory. Thus given a gauge theory, we can always compute the corresponding gravitons and there is a natural equivalence between energies of the gravitons with that of the fields. Note the coupling of field theory with gravity is very weak compared to other gauge couplings (such as strong interaction coupling α_s or the electromagnetic or weak interaction coupling) up to the energy scale relevant for current experiments. Nevertheless there is a naturalness in this correspondence between gauge theory and the geometry it sources, all because of the coupling of the two.

In string theory, a more remarkable and a little less obvious duality arises. Open string excitations are described by a quantum field theory which has scalar and vector fields. In addition, closed string excitations are described by quantum fields which

are tensors of rank two and higher [26][27][28]. Thus open string quantization can give rise to gauge theories with fermions and spin one bosons whereas closed string quantization can incorporate gravitons. In general there will be interactions between open and closed string modes but it turns out that when the interaction is weak, there is indication that the Hilbert space of the open string modes and the Hilbert space of the closed string modes are identical. Thus the conjecture [23] that open string field theory is dual to closed string field theory in the limit where the modes decouple! This remarkable duality is one the primary motivations for proposing a general correspondence between a gauge theory and theory of gravity - in the limit they decouple from each other. In the following sections we will explore this gauge gravity duality, first for conformal field theories and then for gauge theories with running couplings.

2.1 Conformal Field Theory and AdS Geometry

Gauge theories naturally arise when one studies the excitations of branes. In particular excitations of open strings ending on D branes can be described by supersymmetric Yang-Mills multiplet with vector, spinor and scalar fields [26]. If we consider strings ending on a single D3 brane where the D3 brane is embedded in ten dimensional flat space time, then the gauge group is $U(1)$. The D3 brane fills up four dimensional Minkowski space and can move in six dimensional space with coordinates $x^j, j = 4, \dots, 9$. its world volume has coordinates $x^i, i = 0, 1, 2, 3$ and the massless gauge bosons $A_l(x^i), i, l = 0, \dots, 3$ live on the D3 brane world volume. There are six scalar fields $\phi_j(x^i), j = 4, \dots, 9$ and they describe the oscillation of the position of the D3 brane.

On the other hand if we consider N number of parallel D3 branes, then two ends of a string can lie on different branes or on the same brane. There are N^2 choices to place the endpoints which gives N^2 number of vectors. Thus the excitations of the strings stretching between and ending on the branes is described by a $U(N)$ gauge theory [26] [29] with vector, scalars and spinor fields. Note that one takes the massless

modes of the string excitations which means strings have vanishing length and imply that we have coincident branes. Thus excitations of N coincident D3 branes give rise to $U(N)$ gauge theory living in four dimensional flat space time. It also turns out that the gauge coupling does not run and we have a conformal field theory with $U(N)$ gauge group in four dimensions.

The total action for the collection of branes embedded in the ten dimensional space time can be written as [30]

$$\mathcal{S} = \mathcal{S}_{\text{branes}} + \mathcal{S}_{\text{gravity}} + \mathcal{S}_{\text{int}} \quad (2.1)$$

where $\mathcal{S}_{\text{branes}}$ is the action for theory on the branes, $\mathcal{S}_{\text{gravity}}$ is the action for the background geometry on which the branes have been embedded and \mathcal{S}_{int} is the interaction of the brane theory with gravity i.e the interactions of the gauge fields with gravitons. At low energy, the coupling of the gauge theory with the gravitons is negligible and we can ignore the interaction term to obtain a gauge theory living in flat space and a theory of free low energy gravitons.

Let us analyze the system of branes embedded in this geometry using supergravity which is the low energy limit of string theory. The supergravity solution incorporating the back reaction of the flux sourced by the branes was computed back in the 90's [31]. The metric reads [29]

$$\begin{aligned} ds^2 &= \frac{1}{\sqrt{H}} \left(-\tilde{G}(r) dt^2 + d\vec{x}^2 \right) + \frac{\sqrt{H}}{\tilde{G}(r)} dr^2 + r^2 \sqrt{H} d\Omega_5^2 \\ H &= 1 + L^4/r^4, \quad \tilde{G}(r) = 1 - \frac{\tilde{r}_0^4}{r^4}, \quad L^4 = 4\pi g_s N \alpha'^2 \end{aligned} \quad (2.2)$$

where $d\Omega_5^2$ is the metric of five dimensional sphere S^5 , g_s is the string coupling and $\alpha' = l_s^2$ is the string scale. Here horizon is located at $r = \tilde{r}_0$ and extremal limit is achieved by taking $\tilde{r}_0 = 0$. In the extremal limit, we see that we have an horizon at $r = 0$ and for the near horizon limit, $r \ll L$, the above metric takes the following form

$$ds^2 = \frac{r^2}{L^2} \left(-dt^2 + d\vec{x}^2 \right) + \frac{L^2}{r^2} dr^2 + L^2 d\Omega_5^2 \quad (2.3)$$

which is the metric of $AdS_5 \times S_5$, where the AdS throat radius given by L . For an observer located at the boundary $r = \infty$, gravitons from the near horizon region appear to be of low energy as they are red-shifted. As the energy of the gravitons should be measured by an observer at the boundary, low energy modes consist of the gravitons coming from the near horizon region i.e. $AdS_5 \times S_5$ geometry and low energy gravitons from the bulk.

Thus we have two descriptions of the same system of brane configuration at low energy; one in terms of branes and low energy gravitons, the other in terms of $AdS_5 \times S_5$ geometry and low energy gravitons. We can identify the two descriptions and conclude that the brane theory described by $\mathcal{S}_{\text{branes}}$ has an equivalent description in terms of gravitons of $AdS_5 \times S_5$ geometry. Thus the gauge theory on the branes which is a Conformal field theory in flat four dimensional space time is dual to Anti de-Sitter geometry [23]. This duality is known as the AdS/CFT correspondence.

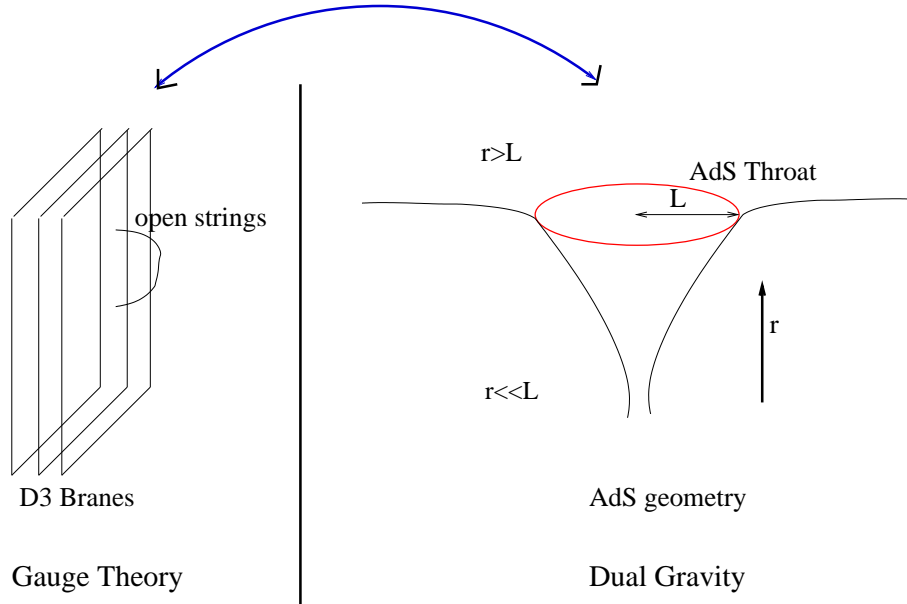


Figure 2.1: Open string gauge theory from D3 branes and its dual geometry.

One may ask at what regime of couplings constant space is this duality valid? To answer this, first note that the string coupling g_s which describes the coupling between the strings and thus the coupling between the gauge fields, can be identified with the

Yang-Mills coupling $g_{\text{YM}}^2 = 4\pi g_s$. On the other hand the 't Hooft coupling for $U(N)$ gauge theory is $\lambda = g_{\text{YM}}^2 N$ and thus the AdS throat radius $L^4 = g_{\text{YM}}^2 N \alpha'^2 = \lambda \alpha'^2$ in α' units is the 't Hooft coupling. The supergravity solution in (2.2,2.3) is only valid when curvature of space is small. Observing that the Ricci scalar for AdS_5 and S_5 are of $\mathcal{O}(L^{-2})$, we conclude that for the solution in (2.2,2.3) to be valid, we need L to be very large. Keeping in mind that length is measured in string units α' , this means $g_s N = \lambda/4\pi \gg 1$. We also want to keep $g_s \ll 1$ and still have $g_s N \gg 1$ is to take $N \rightarrow \infty$. Thus in these limits, we have large N strongly coupled gauge theory with 't Hooft coupling $\lambda \gg 1$ dual to Anti de-Sitter geometry with small curvature. Hence strongly coupled Conformal field theory is dual to weakly coupled Anti de-Sitter gravity.

2.2 Non Conformal Field Theory and Dual Geometry

The duality between conformal field theory and AdS geometry is a particular example of the Holographic Principle mapping Hilbert spaces of gauge theories with that of gravity. By studying brane excitations in various background geometries one can attempt to generalize the AdS/CFT correspondence to include theories with running couplings. In the following sections we first analyze the gauge theories that arise from excitations of branes placed in geometries with conical singularity and then describe their weakly coupled dual gravity.

2.2.1 Gauge Theory from Brane Configuration

We will consider branes placed in geometries with conifolds and study their excitations. Before going into the brane setup, we briefly discuss the conifold geometry. For details, consult [32]-[38]

Consider a six dimensional cone with base $T^{1,1}$ and radial coordinate r . This cone

is a manifold with conic singularity i.e a conifold which is a solution to the Einstein equations in vacuum and it has the metric

$$ds_6^2 = dr^2 + r^2 ds_{T^{1,1}}^2 \quad (2.4)$$

The metric of the base $T^{1,1}$ is given by

$$ds_{T^{1,1}}^2 = \frac{1}{9} \left[d\psi + \sum_{i=1}^2 \cos\theta_i d\phi_i \right]^2 + \frac{1}{6} \sum_{i=1}^2 \left[d\theta_i^2 + \sin^2\theta_i d\phi_i^2 \right] \quad (2.5)$$

where ψ is an angular coordinate ranging from 0 to 4π and (θ_1, ϕ_1) , (θ_2, ϕ_2) parameterizes the two two-spheres S^2 's. The form of the metric makes it clear that $T^{1,1}$ is a $U(1)$ bundle over $S^2 \times S^2$ and has the topology of $S^2 \times S^3$.

Recall that a two dimensional cone embedded in three dimensional space has the familiar embedding equation

$$x^2 + y^2 = z^2$$

where the base of the cone is a circle S^1 with radius $z^2 = r^2$. The six dimensional conifold is a generalization to higher dimensions and has the embedding

$$z_1 z_2 - z_3 z_4 = 0 \quad (2.6)$$

in C^4 , where z_i are complex coordinates given by [32]

$$\begin{aligned} z_1 &= r^{3/2} e^{i/2(\psi - \phi_1 - \phi_2)} \sin(\theta_1/2) \sin(\theta_2/2) \\ z_2 &= r^{3/2} e^{i/2(\psi + \phi_1 + \phi_2)} \cos(\theta_1/2) \cos(\theta_2/2) \\ z_3 &= r^{3/2} e^{i/2(\psi + \phi_1 - \phi_2)} \cos(\theta_1/2) \sin(\theta_2/2) \\ z_4 &= r^{3/2} e^{i/2(\psi - \phi_1 + \phi_2)} \sin(\theta_1/2) \cos(\theta_2/2) \end{aligned} \quad (2.7)$$

To see the symmetries of the space, we can parameterize the conifold with another set of complex coordinates w_i given by

$$\begin{aligned} z_1 &= w_1 + iw_2, & z_2 &= w_1 - iw_2 \\ z_3 &= -w_3 + iw_4, & z_4 &= -w_3 - iw_4 \end{aligned} \quad (2.8)$$

In terms of w_i coordinates, the conifold equation (2.6) becomes

$$\sum w_i^2 = 0 \quad (2.9)$$

On the other hand, the base $T^{1,1}$ is the intersection of the cone with surface

$$\sum |w_i|^2 = r^3 \quad (2.10)$$

We note from (2.9), (2.10) that $T^{1,1}$ is invariant under rotation of the four w_i coordinates, that is under the group $SO(4) \simeq SU(2) \times SU(2)$ and an overall phase rotation. Thus the symmetry group of $T^{1,1}$ is $SU(2) \times SU(2) \times U(1)$. This will come in handy shortly.

With the understanding of the symmetries of our conifold geometry, consider embedding N D3 branes in ten dimensional manifold with the metric

$$ds_{10}^2 = -dt^2 + d\vec{x}^2 + ds_6^2 \quad (2.11)$$

where ds_6^2 is given by (2.4). That is we have four dimensional Minkowski space along with the six dimensional conifold. The D3 branes live in the flat four dimensional space and are placed at the tip of the conifold at fixed radial location $r = 0$, as shown in Fig. 2.2.

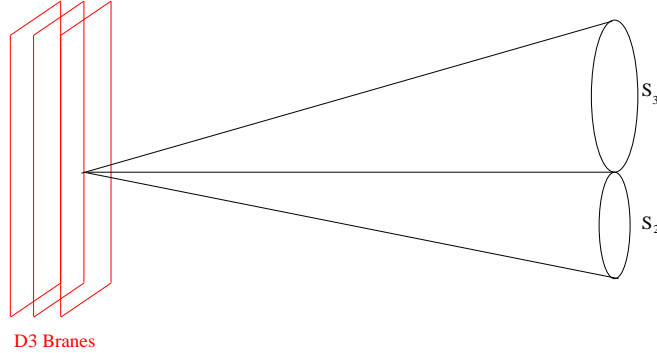


Figure 2.2: N D3 branes placed on conifold singularity.

The excitations of the massless open strings ending on these D3 branes are described by gauge fields and complex matter fields A_i, B_i , $i = 1, 2$ which transform as

bi-fundamental fields under the gauge group $SU(N) \times SU(N)$. Note that the matter fields A_1, A_2 transform under global $SU(2)$ and so do B_1, B_2 under another $SU(2)$ and we also have global $U(1)$ phase rotation. Thus we have $SU(2) \times SU(2) \times U(1)$ global symmetry, which is also the symmetry of the conifold! This is not surprising as these fields describe motion of the D3 branes [26] and the branes move in the conifold direction. Thus the fields A_i, B_i are really coordinates-ordinates of the conifold and can be written as

$$\begin{aligned} z_1 &= A_1 B_1, & z_2 &= A_2 B_2 \\ z_3 &= A_1 B_2, & z_4 &= A_2 B_1 \end{aligned} \quad (2.12)$$

Having analyzed the global symmetry, we would like to understand the origin of the gauge group $SU(N) \times SU(N)$ and the bi-fundamental nature of the matter fields. The nature of the gauge group becomes clear once we analyze the T dual setup of the Type IIB brane configuration in conifold geometry. We will be brief on what follows and for detailed discussion on T dual of Type IIB which is Type IIA brane constructions, please consult [39]-[44].

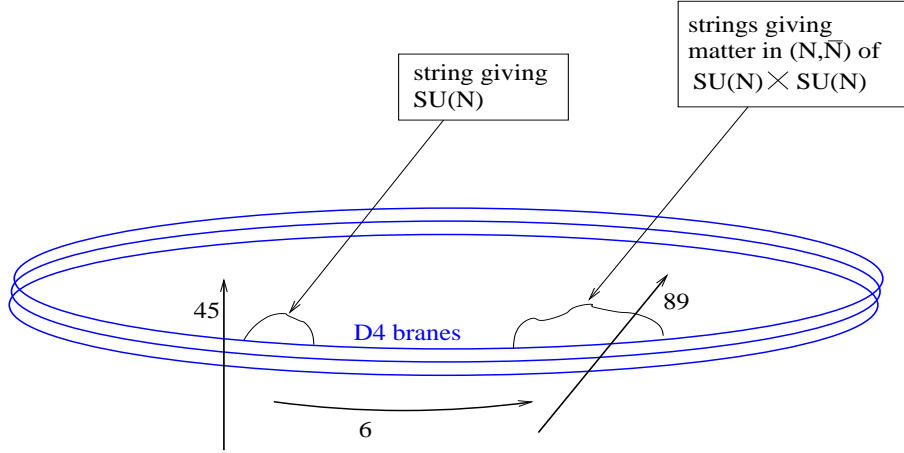


Figure 2.3: T dual of KW model.

Under T duality the cone becomes two intersecting NS5 branes extending along the 012345 and 012389 directions, while the D3 branes become D4 branes along 01236 directions [40]. Here 0123 are the four Minkowski directions, 6 is the radial direction

and 45789 are the angular directions. The T dual setup is shown in Fig **2.3** where we have suppressed 0123 and the 7 direction. As D4 branes can end on NS5 branes, they get divided into two branches between the NS5 branes along the 6 direction as shows in Fig **2.3**. On each branch there are N D4 branes giving rise to $SU(N)$ gauge group. But there are also strings with one end on a D4 brane on one side of the NS5 and the other on the other side NS5 brane, as in Fig **2.3**. These strings give rise to matter fields which transform as (N, \bar{N}) of $SU(N) \times SU(N)$ gauge group. Thus D3 branes at the tip of the conifold gives $SU(N) \times SU(N)$ gauge theory and this is known as the Klebanov-Witten (KW) model [33].

With the understanding of the bi-fundamental nature of the gauge group, we now analyze the various couplings of the KW model. There are three couplings namely the gauge couplings g_1, g_2 corresponding to the gauge group $SU(N) \times SU(N)$ and the coupling h to quartic super potential [33, 46]. The beta functions for the couplings which govern how they change with energy scale μ is given by

$$\beta_{g_1} = -\frac{g_1^3 N}{16\pi^2} \left(\frac{1 + 2\gamma_0}{1 - \frac{g_2^2 N}{8\pi^2}} \right), \quad \beta_{g_2} = -\frac{g_2^3 N}{16\pi^2} \left(\frac{1 + 2\gamma_0}{1 - \frac{g_2^2 N}{8\pi^2}} \right), \quad \beta_\eta = \eta(1 + 2\gamma_0) \quad (2.13)$$

where $\eta = h\mu$ is the dimensionless coupling of the theory. Observe that all the fields in the theory have the same anomalous dimension $\gamma_0(g_1, g_2, h)$ [33, 46] and the three beta functions vanish exactly when

$$\gamma_0(g_1, g_2, h) = -\frac{1}{2} \quad (2.14)$$

which is one equation for the three couplings. The couplings flow with scale μ and once they reach a value such that (g_1, g_2, h) satisfy equation (2.14), there is no flow in the theory. The theory reaches a fixed point and all the couplings stay at that value as energy scale is changed thereafter i.e. the field theory becomes conformal. As equation (2.14) is an equation of a surface in three dimensions, the fixed points in this theory form a *two-dimensional* surface in the three-dimensional space of couplings (see Fig **2.4** for details). The flow of the couplings with scale is of course the Renormalization Group (RG) flow as illustrated in Fig **2.5** below. Since the sign of the two beta

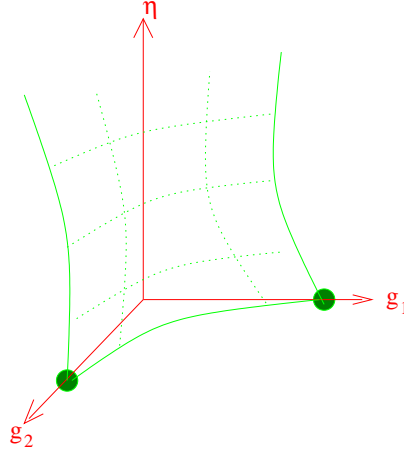


Figure 2.4: The two-dimensional RG surface in the Klebanov-Witten theory.

functions are negative any arbitrary flow in the coupling constant space brings us to the fixed point surface. Thus the brane setup in Fig. **2.2** leads to a conformal field theory with gauge group $SU(N) \times SU(N)$.

Now consider embedding M D5 branes in the conifold geometry with D3 branes where the D5 branes wrap the two cycle S^2 of the conifold base and extend in four Minkowski directions. The D5 branes will slide down to the tip at $r=0$ and we have the brane setup of Fig.**2.6**. We again have bi-fundamental matter fields A_i, B_i with global symmetry group $SU(2) \times SU(2) \times U(1)$. But now due to the additional D5 branes wrapping one of the S^2 's, the gauge group is $SU(N + M) \times SU(N)$.

The bi-fundamental nature of the fields and the different size the gauge groups become clear when we go to the T dual picture as shown in fig. **2.7**. The M D5 branes which wrap only one of the S^2 's (albeit of vanishing radius) become D4 branes which stretch between the NS5 branes only on one branch and thus contribute to only one of the gauge groups [41]. This effectively gives $N + M$ D4 branes stacked in one branch between the NS5 branes where the other branch has N D4 branes. This gives rise to the group $SU(N + M) \times SU(N)$.

As $M \neq 0$ we can define $kM \equiv N + M$ and then the gauge theory is $SU(kM) \times SU((k - 1)M)$. The gauge couplings are now g_k, g_{k-1} for the two gauge groups respectively and we also have η to be the dimensionless coupling of the quartic super

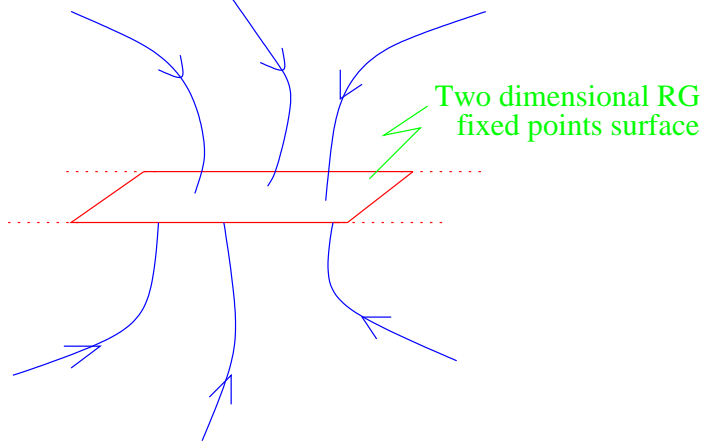


Figure 2.5: The typical RG flows in the Klebanov-Witten theory.

potential. The three beta functions now are [45, 46]:

$$\begin{aligned}\beta_k &= -\frac{g_k^3 k M}{16\pi^2} \left[\frac{(1 + 2\gamma_0) + \frac{2}{k}(1 - \gamma_0)}{1 - \frac{g_k^2 k M}{8\pi^2}} \right], & \beta_\eta &= \eta(1 + 2\gamma_0) \\ \beta_{k-1} &= -\frac{g_{k-1}^3 (k-1) M}{16\pi^2} \left[\frac{(1 + 2\gamma_0) - \frac{2}{k-1}(1 - \gamma_0)}{1 - \frac{g_{k-1}^2 (k-1) M}{8\pi^2}} \right]\end{aligned}\quad (2.15)$$

from which we see that they differ from (2.13) by $\mathcal{O}(1/k)$ factors. Note that there is no point in the coupling constant space where all the three beta functions vanish exactly and thus the theory has no conformal fixed points (except when all couplings are zero and we have free theory at *all scales*).

Solutions to (2.15) along with boundary conditions determine the RG flow of theory and the flow in the space of coupling constants is depicted by the arrows in Fig 2.8. From (2.15) we see that the gauge couplings run logarithmically with scale μ and when both g_k, g_{k-1} are not zero, the difference between them grows with scale. This is the Klebanov-Strassler (KS) model [45]. As the QCD coupling α_s also runs logarithmically with scale, the gauge theory we obtained above could have common features with QCD. Furthermore, as we shall see in the next section, this gauge theory has a dual gravity description and we may learn about strongly coupled QCD by analyzing the dual geometry of the brane setup of Fig 2.6. But before making connections to QCD, let's first analyze the RG flow of KS model.

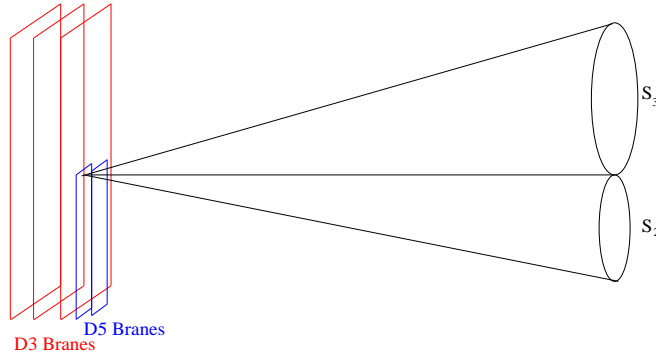


Figure 2.6: D3-D5 system placed at conifold singularity.

The RG flow incorporates a Seiberg duality cascade, where under a series of dualities higher rank gauge groups have equivalent description in terms of lower rank groups. Here we will briefly review the Seiberg duality cascade and its realization in the KS model. For a comprehensive review please consult [46, 47].

Seiberg duality states that strongly coupled $SU(\mathcal{N})$ gauge theory with N_f flavors is dual to weakly coupled $SU(N_f - \mathcal{N})$ gauge theory with N_f flavors. Recall that we have $SU(N + M) \times SU(N)$ gauge group and the $SU(N + M)$ branch has $2N$ effective flavors while the $SU(N)$ branch has $2(N + M)$ flavors [45, 46]. To see this, consider the fields A_1, A_2 and note that each of them have color indices under two color groups $SU(N + M)$ and $SU(N)$. If one fixes the color under one of the group, say $SU(N + M)$, then the color indices for the group $SU(N)$ can be thought of as flavor indices. Then for the $SU(N + M)$ color symmetry group, the $SU(N)$ group appears as flavor symmetry group. As we have two fields A_1 and A_2 , this means that the $SU(N + M)$ branch has $2N$ effective flavors. The same argument shows that $SU(N)$ group has $2(N + M)$ effective flavors.

Now consider the flow on the $\eta = 0$ plane given by curve 2 of Fig 2.8. The coupling g_{k-1} corresponding to gauge group $SU(N)$ shrinks while g_k corresponding to $SU(N + M)$ grows as scale is changed from UV to IR. . At the end point of curve 2, the $SU(N + M)$ gauge group is strongly coupled with $2N$ effective flavors and thus

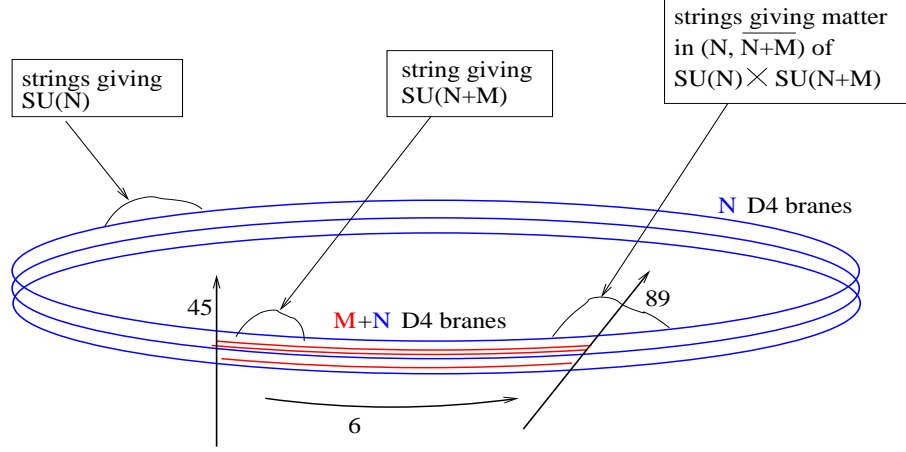


Figure 2.7: T dual of KS model.

it is dual to weakly coupled $SU(2N - (N + M)) = SU(N - M)$ gauge theory under Seiberg duality. This means the end point of curve 2 has two equivalent descriptions; one in terms of $SU(N + M)$ strongly coupled gauge theory with $2N$ flavors and the other in terms of $SU(N - M)$ weakly coupled gauge theory with $2N$ flavors. Thus $SU(N + M) \times SU(N)$ gauge theory is dual to $SU(N - M) \times SU(N)$ gauge theory. But $N - M = (k - 2)M$ and thus we can draw the endpoint of curve 2 on another coupling constant space with couplings g_{k-1}, g_{k-2} as shown in Fig 2.9. In fact the RG flow of curve 3 can be identified with that of curve 1 in the space of new couplings g_{k-1}, g_{k-2} and new quartic coupling $\tilde{\eta}$. The RG flow of the new couplings $g_{k-1}, g_{k-2}, \tilde{\eta}$ can be obtained by replacing k with $k - 1$ and η by $\tilde{\eta}$ in (2.15). Solving the RG equations, one observe that when g_k of $SU(N + M)$ sector grows, g_{k-2} of $SU(N - M)$ shrinks, so the flows of curve 3 in Fig 2.9(a) and of curve 1 in Fig 2.9(b) are in opposite direction. This also means $\eta \sim 1/\tilde{\eta}$ and the identification of curve 3 with curve 1 is shown in Fig 2.9 is justified.

Denoting $N - M = \tilde{N}$, we now have $SU(\tilde{N} + M) \times SU(\tilde{N})$ and repeating the same arguments as before we can perform another Seiberg duality to obtain the $SU(\tilde{N} - M) \times SU(\tilde{N})$ gauge group. At each step of the duality, the effective \bar{N} (defined from gauge group $SU(\bar{N} + M) \times SU(\bar{N})$) is reduced by M units and we have a cascade of dualities known as the Seiberg duality cascade. The reduction continues

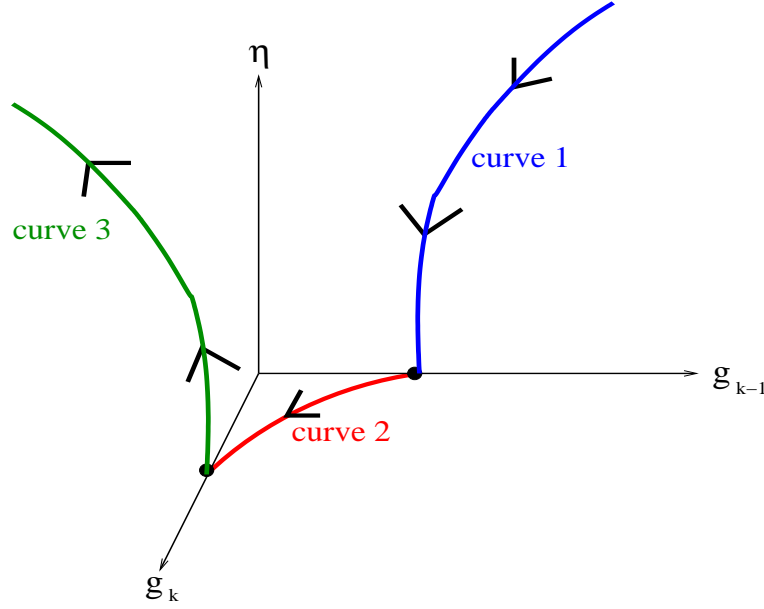


Figure 2.8: RG flow of KS model.

until we reach a point when one of the groups has zero size and we end up with $SU(0) \times SU(M) = SU(M)$ gauge group.

By suppressing the quartic coupling $\eta, \tilde{\eta}$, we can draw the RG flows of the couplings $(g_k, g_{k-1}, \dots, g_{k-n})$ on the same plot as depicted on Fig **2.10**. Each line of the boundary is the axis for coupling $g_{\tilde{k}}$ and its Seiberg dual $g_{\tilde{k}-2}$. This depiction is indeed valid as when $g_{\tilde{k}}$ grows and becomes strong, its dual $g_{\tilde{k}-2}$ shrinks and becomes weak and vice-versa. Intersecting lines describe gauge couplings $(g_{\tilde{k}}, g_{\tilde{k}-1})$ and we move from one pair of couplings to the other with the flow. The cut in the Fig **2.10** indicates that the flow do not take us back to the original coupling g_{k-1} , rather the flow takes us to another coupling g_{k-n} and we move from one sheet of coupling constant space to another.

Thus we see that in the KS model there are multiple equivalent descriptions of the same gauge theory and each description is related to the other by Seiberg duality. As we go from UV to IR, we can describe the gauge theory with lower and lower rank groups. In the language of Wilsonian RG flow, we are integrating out UV modes and obtaining effective Lagrangians for lower and lower rank gauge groups in the

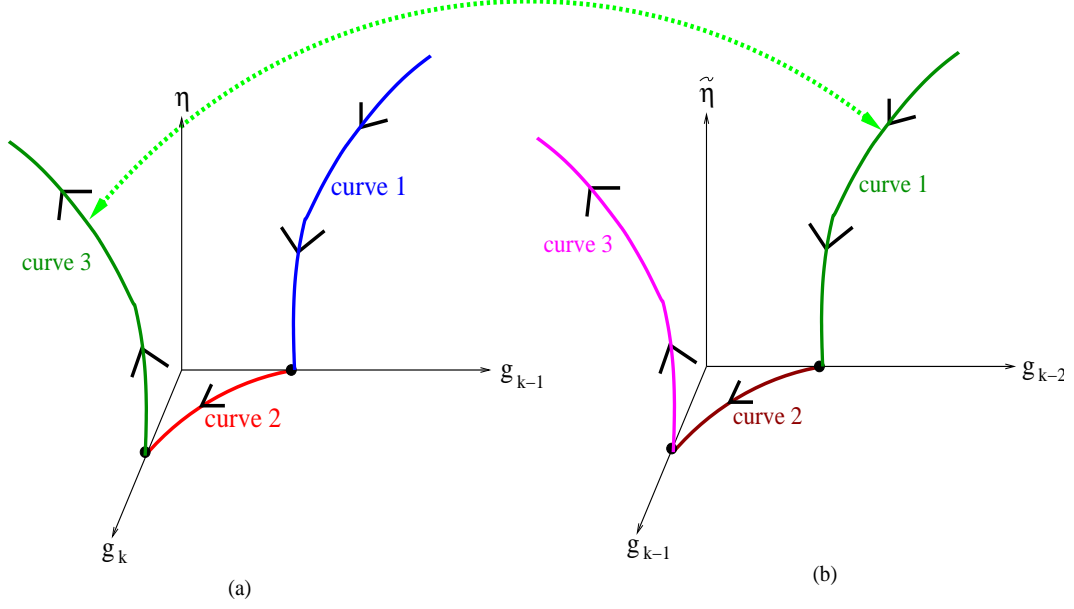


Figure 2.9: RG flow of $SU(N+M) \times SU(N)$ along with its dual $SU(N-M) \times SU(N)$

IR. If we keep all the relevant, irrelevant and marginal operators as we integrate out, physical observables of course should not change with the RG flow. But if we leave out certain operators to obtain the effective Lagrangians, the flow will result in changes in physical observables. We will have more to say about this subtlety in the coming sections.

In summary, the gauge theory in the KS model has a very rich structure with the rank of the gauge group growing in the UV. In fact the far UV of gauge theory is Seiberg dual to an infinite rank gauge group while the far IR is a rather simple $SU(M)$ gauge theory. But we still do not have matter in the fundamental representation and to make connection with QCD, we need to amend the KS brane setup.

In order to obtain fundamental matter, one introduces D7 branes. The D7 branes can be embedded in various ways [38, 48, 49, 50, 51, 52, 53] and in particular the D7 brane world volume will depend on the background geometry along with the boundary conditions. In Ouyang's model [38] seven branes are embedded via the following equation (see also [54]):

$$z \equiv r^{\frac{3}{2}} \exp \left[\frac{i(\psi - \phi_1 - \phi_2)}{2} \right] \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} = \mu \quad (2.16)$$

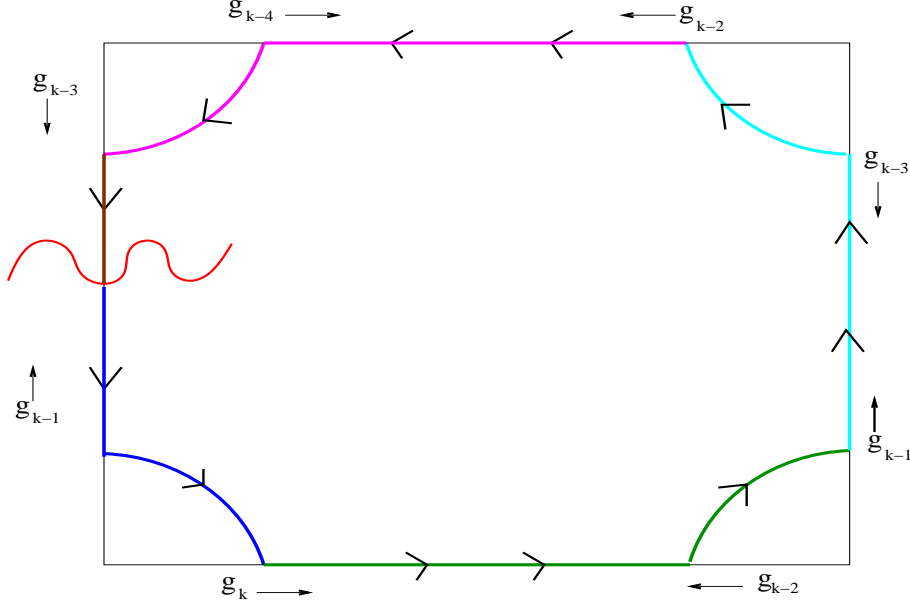


Figure 2.10: RG flow of the gauge theory as it cascades under Seiberg duality.

where μ is a complex quantity. In the limit where $\mu \rightarrow 0$, the seven branes are oriented along two branches:

$$\begin{aligned} \text{Branch 1 : } & \theta_1 = 0, \quad \phi_1 = 0 \\ \text{Branch 2 : } & \theta_2 = 0, \quad \phi_2 = 0 \end{aligned} \quad (2.17)$$

From the above observe that the seven branes in branch 1 wrap a four cycle (θ_2, ϕ_2) and (ψ, r) in the internal space and is stretched along the space time directions (t, x, y, z) . Similarly in seven branes on branch 2 would wrap a four-cycle $(\theta_1, \phi_1, r, \psi)$.

With the above embedding one needs to check whether Gauss's law is violated. As the seven branes wrap a non-compact four cycle filling the entire r direction, the field lines sourced by the branes extend only in the compact directions. If we draw a Gaussian surface, all the field lines that go out of the surface come back into the surface as the space is compact. This means there cannot be any net charge due to the seven branes. This paradox can be resolved by allowing the seven brane to wrap a topologically trivial cycle so that it can end *abruptly* at some $r = r_{\min}$ when the embedding is (2.16). Thus the D7 brane extend in the radial direction with $r_{\min} < r < \infty$ which is similar to the seven brane configuration of [48, 53]. Of course

there are N_f D7 branes which can overlap and the final configuration of branes is sketched in Fig **2.11**. In section **2.4** we will discuss modification of the embedding (2.16) which will be more relevant in describing large N QCD.

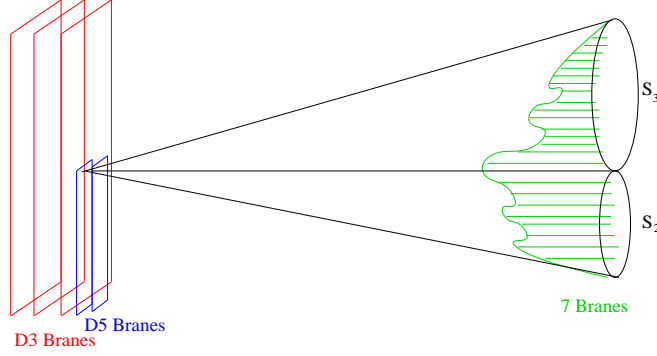


Figure 2.11: D3-D5 branes along with D7 branes embedded in conifold geometry.

In addition to the bi-fundamental fields A_i, B_i , introduction of the D7 branes give rise to flavor symmetry group $SU(N_f) \times SU(N_f)$ and matter fields $q, \tilde{q}, Q, \tilde{Q}$ which transform as fundamental under the gauge group $SU(N+M) \times SU(N)$ [38]. The bi-fundamental nature of the fields follows from the same T duality argument as before and the beta functions now take the form:

$$\begin{aligned} \beta_k &= -\frac{g_k^3 k M}{16\pi^2} \left[\frac{(1+2\gamma_0) + \frac{2}{k}(1-\gamma_0)}{1 - \frac{g_k^2 k M}{8\pi^2}} + \frac{N_f(1-2\gamma_q)}{kM} \right], \quad \beta_\eta = \eta(1+2\gamma_0) \\ \beta_{k-1} &= -\frac{g_{k-1}^3 (k-1) M}{16\pi^2} \left[\frac{(1+2\gamma_0) - \frac{2}{k-1}(1-\gamma_0)}{1 - \frac{g_{k-1}^2 (k-1) M}{8\pi^2}} + \frac{N_f(1-2\gamma_q)}{(k-1)M} \right] \end{aligned} \quad (2.18)$$

where γ_q is the anomalous dimension of the q field. Again we have logarithmic running of the couplings but also matter which transforms as fundamental of the gauge group. In Table **2.1**, we list the various matter fields and their representation under local and global symmetry groups for the brane setup in Fig **2.11**.

Field	$SU(N+M) \times SU(N)$	$SU(N_f) \times SU(N_f)$	$SU(2) \times SU(2)$
q	$(\mathbf{N} + \mathbf{M}, 1)$	$(\mathbf{N}_f, 1)$	$(\mathbf{1}, \mathbf{1})$
\tilde{q}	$(\overline{\mathbf{N} + \mathbf{M}}, 1)$	$(1, \mathbf{N}_f)$	$(\mathbf{1}, \mathbf{1})$
Q	$(1, \mathbf{N} + \mathbf{M})$	$(\overline{\mathbf{N}_f}, 1)$	$(\mathbf{1}, \mathbf{1})$
\tilde{Q}	$(1, \overline{\mathbf{N} + \mathbf{M}})$	$(1, \overline{\mathbf{N}_f})$	$(\mathbf{1}, \mathbf{1})$
$A_{1,2}$	$(\mathbf{N} + \mathbf{M}, \overline{\mathbf{N} + \mathbf{M}})$	$(\overline{\mathbf{N}_f}, \mathbf{N}_f)$	$(\mathbf{2}, \mathbf{1})$
$B_{1,2}$	$(\overline{\mathbf{N} + \mathbf{M}}, \mathbf{N} + \mathbf{M})$	$(\mathbf{N}_f, \overline{\mathbf{N}_f})$	$(\mathbf{1}, \mathbf{2})$

Table 2.1: The field content and their representation under symmetry groups.

The field theory here also cascades to lower and lower rank gauge groups under Seiberg duality. But now due to the presence of fundamental flavor, the effective flavor for $SU(N+M)$ strongly coupled gauge group is $2N + N_f$ and its Seiberg dual weakly coupled theory is $SU(2N + N_f - N - M) = SU(N - M + N_f)$. This means that at the i th step of the cascade, the effective \bar{N} (again defined from the gauge group $SU(\bar{N} + M) \times SU(\bar{N})$) is reduced by $(M - kN_f)$ units, with k a natural number. This reduction continues until we end up with just one group $SU(0) \times SU(M - jN_f) = SU(M - jN_f)$. Here j is the number of dualities performed starting with gauge group $SU(N+M) \times SU(N)$ at some UV scale. This brane setup in Fig 2.11 is the Ouyang embedding of D7 branes on Klebanov-Strassler model and we will refer to it as the Ouyang-Klebanov-Strassler (OKS) model.

In summary, for OKS model we have a field theory which is non conformal with matter in fundamental representation with N_f flavors and the gauge couplings run logarithmically with scale. At some UV scale the gauge group has description in terms of $SU(N+M) \times SU(N)$ group while at the far IR we end up with $SU(M - jN_f)$ gauge group.

For the gauge theories arising from brane configurations in Fig 2.2, 2.6 and 2.11, we would like to compute matrix elements as they are physical quantities of interest. Note that it is the t'Hooft couplings $\lambda_k = (N+M)g_k^2(\Lambda)$ and $\lambda_{k-1} = Ng_{k-1}^2(\Lambda)$ that are relevant for computing propagators and scattering amplitudes. At a scale when either of the two t'Hooft couplings λ_k, λ_{k-1} become large, perturbative methods fail

and we need to resort to other techniques. One approach is to study the theory on the lattice while the other is to find the dual gravity. We will discuss the latter in the following section.

2.2.2 The Dual Gravity of the Brane Theory

Having discussed in some detail the properties of gauge theories that arise from branes in conifold geometry, we will now analyze the same brane configurations at low energy using supergravity (SUGRA) approximation of string theory. Type IIB supergravity action including local sources in ten dimensions is [55][56]:

$$\begin{aligned}
S_{\text{total}} &= S_{\text{SUGRA}} + S_{\text{loc}} = \frac{1}{2\kappa_{10}^2} \int d^{10}x \sqrt{G} \left(R + \frac{\partial_\mu \bar{\tau} \partial^\mu \tau}{2|\text{Im}\tau|^2} - \frac{1}{2} |\tilde{F}_5|^2 - \frac{G_3 \cdot \bar{G}_3}{12\text{Im}\tau} \right) \\
&+ \int_{\Sigma_8} C_4 \wedge R_{(2)} \wedge R_{(2)} + \frac{1}{8i\kappa_{10}^2} \int \frac{C_4 \wedge G_3 \wedge \bar{G}_3}{\text{Im}\tau} + S_{\text{loc}}
\end{aligned} \tag{2.19}$$

in Einstein frame, where $\tau = C_0 + ie^{-\phi}$ is the axio-dilaton with C_0 being the axion and ϕ the dilaton field and \tilde{F}_5 is the five-form flux sourced by the D3 branes. Here $G_3 \equiv F_3 - \tau H_3$ with F_3 the RR three form flux sourced by D5 branes and $H_3 = dB_2$ the NS-NS three form flux with B_2 being the NS-NS two form. We also have C_4 the four-form potential, $G = \sqrt{\det G_{\mu\nu}}$ with $G_{\mu\nu}$ being the metric, $R_{(2)}$ the curvature two-form, and S_{loc} is the action for localized sources in the system (i.e. D7 branes and other local sources that we may consider). The above action (2.19) is the most general supergravity action one gets by placing branes in various geometries and we will consider the relevant terms for the the brane setups of Fig **2.2**, **2.6** and **2.11** separately.

First consider the brane setup of Fig **2.2**. The N D3 branes sources Ramond-Ramond (RR) five form flux in the supergravity action but do not source three form flux and neither the axio-dilaton field. Minimizing the action (2.19) with the only $\tilde{F}_5 \neq 0$ being the source, we get the following metric [33]

$$ds^2 = \frac{1}{\sqrt{H}} \left(-dt^2 + d\vec{x}^2 \right) + \sqrt{H} dr^2 + r^2 \sqrt{H} ds_{T^{1,1}}^2$$

$$\begin{aligned}
H &= 1 + h, \quad h = \frac{L^4}{r^4}, \quad L^4 = 4\pi g_s N \alpha'^2 \\
g_s F_5 &= d^4 x \wedge dh^{-1} + * (d^4 x \wedge dh^{-1})
\end{aligned} \tag{2.20}$$

where $ds_{T^{1,1}}^2$ is the metric of $T^{1,1}$ given by (2.5), $*$ is the hodge star operator and we have taken the extremal limit. The horizon is located at $r = 0$ and the near horizon limit ($r \ll L$) of the above metric (2.20) is that of $AdS_5 \times T^{1,1}$. Using the same arguments as in section **2.1**, the near horizon gravitons appear to be of low energy as measured by an observer at the boundary $r = \infty$. On the other hand at low energy, the $SU(N) \times SU(N)$ gauge theory that lives on D3 brane world volume i.e on four dimensional space, decouples from gravity. Thus we conclude that $SU(N) \times SU(N)$ gauge theory in four dimensional flat space is dual to $AdS_5 \times T^{1,1}$ geometry.

But in what regime is this duality valid? For our supergravity solution to hold, we need $L \gg 1$ in α' units, which means that $g_s N \gg 1$. As we want to ignore effects of string loops in supergravity action, we need $g_s \ll 1$ and combined with the requirement $g_s N \gg 1$, this means $N \rightarrow \infty$. On the other hand the gauge couplings g_1, g_2 corresponding to the gauge group $SU(N) \times SU(N)$ is related to the dilaton and B_2 appearing on SUGRA action by [33][41]

$$\begin{aligned}
\frac{8\pi^2}{g_1^2} &= e^{-\Phi} \left[\pi + \frac{1}{2\pi} \left(\int_{S^2} B_2 \right) \right] \\
\frac{8\pi^2}{g_2^2} &= e^{-\Phi} \left[\pi - \frac{1}{2\pi} \left(\int_{S^2} B_2 \right) \right]
\end{aligned} \tag{2.21}$$

For brane configuration in Fig **2.2**, the dual geometry has no B_2 and dilaton is constant with its value set by the string coupling i.e. $e^{-\phi} = 1/g_s$ which gives $g_1^2 = g_2^2 = 8\pi g_s$. Thus from dual supergravity we conclude that the gauge couplings do not run with scale and we have conformal field theory with large t'Hooft coupling $\lambda_1 = \lambda_2 = 8\pi g_s N \gg 1$. This is exactly consistent with our analysis of the gauge theory of KW model in the previous section where we found that the gauge couplings always reach conformal fixed point surface of Fig **2.5**.

The field theories arising from the brane setups in Fig **2.2**, **2.6** and **2.11** are at zero temperature. Introducing temperature in the field theory boils down to introducing

black holes in the dual geometry. For the KW model, metric for the dual geometry incorporating finite temperature of the field theory is given by

$$\begin{aligned} ds^2 &= -\frac{g}{\sqrt{h}}dt^2 + \frac{1}{h}d\vec{x}^2 + \frac{\sqrt{h}}{g}dr^2 + r^2\sqrt{h}ds_{T^{1,1}}^2 \\ g &= 1 - \frac{r_h^4}{r^4} \end{aligned} \quad (2.22)$$

where r_h is the horizon of the black-hole. Note that the above metric (2.22) can be obtained by minimizing the SUGRA action (2.19) with only $\tilde{F}_5 \neq 0$ and every other source zero. Thus metrics in (2.20) and (2.22) can be obtained from the same action with the same sources.

Next for the brane setups in Fig **2.6** and **2.11**, the most general SUGRA solution takes the form [57]

$$ds^2 = \frac{1}{\sqrt{h}} \left(-g_1 dt^2 + dx^2 + dy^2 + dz^2 \right) + \sqrt{h} \left[g_2^{-1} dr^2 + r^2 d\mathcal{M}_5^2 \right] \quad (2.23)$$

where g_i are functions¹ that determine the presence of the black hole, h is the 10d warp factor that could be a function of all the internal coordinates and $d\mathcal{M}_5^2$ is given by:

$$\begin{aligned} d\mathcal{M}_5^2 = & h_1 (d\psi + \cos \theta_1 d\phi_1 + \cos \theta_2 d\phi_2)^2 + h_2 (d\theta_1^2 + \sin^2 \theta_1 d\phi_1^2) + \\ & + h_4 (h_3 d\theta_2^2 + \sin^2 \theta_2 d\phi_2^2) + h_5 \cos \psi (d\theta_1 d\theta_2 - \sin \theta_1 \sin \theta_2 d\phi_1 d\phi_2) + \\ & + h_5 \sin \psi (\sin \theta_1 d\theta_2 d\phi_1 - \sin \theta_2 d\theta_1 d\phi_2) \end{aligned} \quad (2.24)$$

with h_i being the six-dimensional warp factors. The advantage of writing the background in the above form is that it includes all possible deformations in the presence of seven branes, fluxes and other localized sources in the theory. The difficulty however is that the equations for the warp factors h_i are coupled higher order differential equations which do not have simple analytical solutions. The original KS solution without black hole and seven branes is obtained in the limit

$$h_3 = g_i = 1, \quad h_i = \text{fixed} \quad (2.25)$$

¹They would in general be functions of (r, θ_i) . We will discuss this later.

In the presence of seven branes we obtain:

$$h_5 = 0, \quad h_3 = 1, \quad h_4 - h_2 = a, \quad g_i = 1 \quad (2.26)$$

which puts a seven brane in a resolved conifold background with $a = \text{constant}$ [54] using the so-called Ouyang embedding [38]. For $a = 0$ and the following choice of h_i ,

$$h_1 = \frac{1}{9}, \quad h_2 = h_4 = \frac{1}{6}, \quad h_3 = 1 \quad (2.27)$$

the metric takes the following form [38]

$$\begin{aligned} ds^2 &= \frac{1}{\sqrt{h}} \left(-dt^2 + dx^2 + dy^2 + dz^2 \right) + \sqrt{h} \left[dr^2 + r^2 ds_{T^{1,1}}^2 \right] \\ h &= \frac{L^4}{r^4} \left[1 + \frac{3g_s M^2}{2\pi N} \log r \left\{ 1 + \frac{3g_s N_f}{2\pi} \left(\log r + \frac{1}{2} \right) + \frac{g_s N_f}{4\pi} \log \left(\sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right) \right\} \right] \end{aligned} \quad (2.28)$$

which is Ouyang's solution [38]. In the limit $N_f = 0$, the above metric (2.28) reduces to the metric of Klebanov-Strassler model [45].

As mentioned above, a black hole could be inserted in this background by switching on a non-trivial g_i . However a naive choice of fluxes in (2.26) will break supersymmetry [54]. In general we will not restrict to dual geometries which are supersymmetric even at zero temperature and in particular supersymmetry will be explicitly broken by the introduction of black hole.

Once we introduce black hole, $g_i \neq 1$ and we do not expect h_i to remain constant anymore. We also expect M and N_f in (2.28) to be given by some M_{eff} and N_f^{eff} respectively. Our first approximation would then be to make the following ansatz for the h_i , M_{eff} and N_f^{eff} :

$$\begin{aligned} h_1 &= \frac{1}{9} + \mathcal{O}(g_s), \quad h_2 = h_4 = \frac{1}{6} + \mathcal{O}(g_s), \quad h_3 = 1 + \mathcal{O}(g_s) \\ M_{\text{eff}} &= M + \sum_{m \geq n} a_{mn} (g_s N_f)^m (g_s M)^n, \quad N_f^{\text{eff}} = N_f + \sum_{m \geq n} b_{mn} (g_s N_f)^m (g_s M)^n \end{aligned} \quad (2.29)$$

with a_{mn}, b_{mn} could in principle be functions of the internal coordinates (ψ, ϕ_i, θ_i) . Note that we have made $m \geq n$ in the above expansions because the precise limits

for which our supergravity solution would be valid are:

$$\left(g_s, g_s N_f, g_s^2 M N_f, \frac{g_s M^2}{N} \right) \rightarrow 0, \quad (g_s N, g_s M) \rightarrow \infty \quad (2.30)$$

These limits of the variables bring us closer to the Ouyang solution with little squashing of the two-spheres. This also means that the warp factor h in (2.23) can be written as [57]:

$$h = \frac{L^4}{r^4} \left[1 + \frac{3g_s M_{\text{eff}}^2}{2\pi N} \log r \left\{ 1 + \frac{3g_s N_f^{\text{eff}}}{2\pi} \left(\log r + \frac{1}{2} \right) + \frac{g_s N_f^{\text{eff}}}{4\pi} \log \left(\sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right) \right\} \right] \quad (2.31)$$

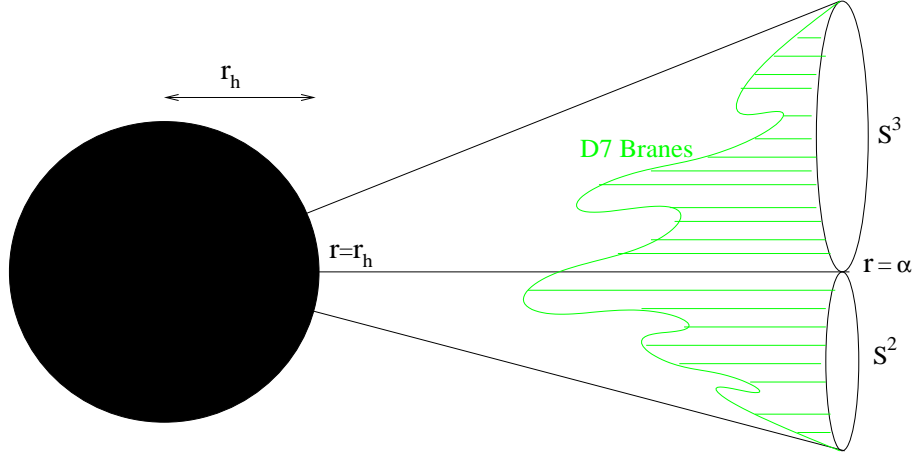


Figure 2.12: Dual geometry of Ouyang-Klebanov-Strassler model with black hole.

Now for the black hole factors, what are the choices for g_i that are consistent with taking back reactions of all the sources in our setup? The Einstein's equations derived from the action (2.19) for the KS and OKS model (where sources are functions of internal coordinates) fixes g_i 's to be functions of (r, θ_1, θ_2) . Our ansatz therefore is [57]:

$$g_1(r, \theta_1, \theta_2) = 1 - \frac{r_h^4}{r^4} + \mathcal{O}(g_s^2 M N_f), \quad g_2(r, \theta_1, \theta_2) = 1 - \frac{r_h^4}{r^4} + \mathcal{O}(g_s^2 M N_f) \quad (2.32)$$

where r_h is the horizon, and the (θ_1, θ_2) dependences come from the $\mathcal{O}(g_s^2 M N_f)$ corrections. The resolution parameter a is no longer a constant but a function of horizon radius and number of D5 and D7 branes i.e. $a = a(r_h) + \mathcal{O}(g_s^2 M N_f)$. In Fig

2.12 we sketched the dual geometry of OKS model with a black hole. The geometry is a resolved deformed conifold with squashing of the two S^2 's and as the local action for D7 branes appear explicitly in (2.19), we have sketched their embedding in the dual geometry. Note that D3 and D5 branes contribute to the geometry through the fluxes but do not explicitly enter into the action. Hence these branes are not sketched in the dual geometry and only D7 branes form the localized sources.

But what are the fluxes in (2.19) that give rise to a metric of the form (2.23) with g_i as in (2.32)? The background RR three and five-form fluxes can be succinctly written as:

$$\begin{aligned}
H_3 &= dr \wedge e_\psi \wedge (c_1 d\theta_1 + c_2 d\theta_2) + dr \wedge (c_3 \sin \theta_1 d\theta_1 \wedge d\phi_1 - c_4 \sin \theta_2 d\theta_2 \wedge d\phi_2) \\
&\quad + \left(\frac{r^2 + 6a^2}{2r} c_1 \sin \theta_2 d\phi_2 - \frac{r}{2} c_2 \sin \theta_1 d\phi_1 \right) \wedge d\theta_1 \wedge d\theta_2, \\
\tilde{F}_3 &= -\frac{1}{g_s} dr \wedge e_\psi \wedge (c_1 \sin \theta_1 d\phi_1 + c_2 \sin \theta_2 d\phi_2) \\
&\quad + \frac{1}{g_s} e_\psi \wedge (c_5 \sin \theta_1 d\theta_1 \wedge d\phi_1 - c_6 \sin \theta_2 d\theta_2 \wedge d\phi_2) \\
&\quad - \frac{1}{g_s} \sin \theta_1 \sin \theta_2 \left(\frac{r}{2} c_2 d\theta_1 - \frac{r^2 + 6a^2}{2r} c_1 d\theta_2 \right) \wedge d\phi_1 \wedge d\phi_2. \tag{2.33}
\end{aligned}$$

where H_3 is closed and $\tilde{F}_3 \equiv F_3 - C_0 H_3$, C_0 being the ten dimensional axion. The derivations of the coefficients appearing in (2.33) are rather involved and can be found in [57]². Here we just quote the results for a constant resolution parameter a :

$$\begin{aligned}
c_1 &= \frac{g_s^2 M N_f}{4\pi r (r^2 + 6a^2)^2} (72a^4 - 3r^4 - 56a^2 r^2 \log r + a^2 r^2 \log(r^2 + 9a^2)) \cot \frac{\theta_1}{2} \\
c_2 &= \frac{3g_s^2 M N_f}{4\pi r^3} (r^2 - 9a^2 \log(r^2 + 9a^2)) \cot \frac{\theta_2}{2} \tag{2.34}
\end{aligned}$$

²Observe that the background EOMs cannot be trivially worked out by solving SUGRA EOMs with fluxes and seven branes sources. This is because, even if we know the energy momentum tensors for the fluxes, the energy momentum tensors for N_f coincident seven branes are not known in the literature. In particular *non-abelian* Born-Infeld action for N_f seven branes on a *curved* background is unknown. In the absence of such direct approach, we use an alternative method to derive the EOMs. This method uses the ISD (imaginary self-duality) properties of the background fluxes and fields. Details on this appears in [38][54] for the case without black hole. For the geometry with a black hole we found [57] that one could find consistent solutions to EOMs using similar arguments.

$$\begin{aligned}
c_3 &= \frac{3g_s Mr}{r^2 + 9a^2} + \frac{g_s^2 MN_f}{8\pi r(r^2 + 9a^2)} \left[-36a^2 - 36r^2 \log a + 34r^2 \log r \right. \\
&\quad \left. + (10r^2 + 81a^2) \log(r^2 + 9a^2) + 12r^2 \log \left(\sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right) \right] \\
c_4 &= \frac{3g_s M(r^2 + 6a^2)}{\kappa r^3} + \frac{g_s^2 MN_f}{8\pi \kappa r^3} \left[18a^2 - 36(r^2 + 6a^2) \log a + (34r^2 + 36a^2) \log r \right. \\
&\quad \left. + (10r^2 + 63a^2) \log(r^2 + 9a^2) + (12r^2 + 72a^2) \log \left(\sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right) \right] \\
c_5 &= g_s M + \frac{g_s^2 MN_f}{24\pi(r^2 + 6a^2)} \left[18a^2 - 36(r^2 + 6a^2) \log a + 8(2r^2 - 9a^2) \log r \right. \\
&\quad \left. + (10r^2 + 63a^2) \log(r^2 + 9a^2) \right] \\
c_6 &= g_s M + \frac{g_s^2 MN_f}{24\pi r^2} \left[-36a^2 - 36r^2 \log a + 16r^2 \log r + (10r^2 + 81a^2) \log(r^2 + 9a^2) \right]
\end{aligned}$$

with $\kappa = \frac{r^2 + 9a^2}{r^2 + 6a^2}$. All the above coefficients have further corrections that we will discuss later. Finally, this allows us to write the NS 2-form potential:

$$\begin{aligned}
B_2 &= \left(b_1(r) \cot \frac{\theta_1}{2} d\theta_1 + b_2(r) \cot \frac{\theta_2}{2} d\theta_2 \right) \wedge e_\psi \\
&+ \left[\frac{3g_s^2 MN_f}{4\pi} \left(1 + \log(r^2 + 9a^2) \right) \log \left(\sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right) + b_3(r) \right] \sin \theta_1 d\theta_1 \wedge d\phi_1 \\
&- \left[\frac{g_s^2 MN_f}{12\pi r^2} \left(-36a^2 + 9r^2 + 16r^2 \log r + r^2 \log(r^2 + 9a^2) \right) \log \left(\sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right) + b_4(r) \right] \\
&\quad \times \sin \theta_2 d\theta_2 \wedge d\phi_2
\end{aligned} \tag{2.35}$$

with the r -dependent functions

$$\begin{aligned}
b_1(r) &= \frac{g_s^2 MN_f}{24\pi(r^2 + 6a^2)} (18a^2 + (16r^2 - 72a^2) \log r + (r^2 + 9a^2) \log(r^2 + 9a^2)) \\
b_2(r) &= -\frac{3g_s^2 MN_f}{8\pi r^2} (r^2 + 9a^2) \log(r^2 + 9a^2)
\end{aligned} \tag{2.36}$$

and $b_3(r)$ and $b_4(r)$ are given by the first order differential equations

$$b'_3(r) = \frac{3g_s Mr}{r^2 + 9a^2} + \frac{g_s^2 MN_f}{8\pi r(r^2 + 9a^2)} \left[-36a^2 - 36a^2 \log a + 34r^2 \log r \right. \\
\left. + (10r^2 + 81a^2) \log(r^2 + 9a^2) \right] \tag{2.37}$$

$$\begin{aligned}
b'_4(r) &= -\frac{3g_s M(r^2 + 6a^2)}{\kappa r^3} - \frac{g_s^2 MN_f}{8\pi \kappa r^3} \left[18a^2 - 36(r^2 + 6a^2) \log a \right. \\
&\quad \left. + (34r^2 + 36a^2) \log r + (10r^2 + 63a^2) \log(r^2 + 9a^2) \right]
\end{aligned} \tag{2.38}$$

Putting back the forms of c_i in (2.33) we can see how exactly the fluxes change with conifold coordinates, but being quite involved, it is hard to compare with the

Ouyang model. However there exist an alternative way to rewrite the fluxes which would tell us exactly how the black hole modifies the original Ouyang setup. This can be presented in the following way:

$$\begin{aligned}
\tilde{F}_3 &= 2M\mathbf{A}_1 \left(1 + \frac{3g_s N_f}{2\pi} \log r\right) e_\psi \wedge \frac{1}{2} (\sin \theta_1 d\theta_1 \wedge d\phi_1 - \mathbf{B}_1 \sin \theta_2 d\theta_2 \wedge d\phi_2) \\
&\quad - \frac{3g_s M N_f}{4\pi} \mathbf{A}_2 \frac{dr}{r} \wedge e_\psi \wedge \left(\cot \frac{\theta_2}{2} \sin \theta_2 d\phi_2 - \mathbf{B}_2 \cot \frac{\theta_1}{2} \sin \theta_1 d\phi_1 \right) \\
&\quad - \frac{3g_s M N_f}{8\pi} \mathbf{A}_3 \sin \theta_1 \sin \theta_2 \left(\cot \frac{\theta_2}{2} d\theta_1 + \mathbf{B}_3 \cot \frac{\theta_1}{2} d\theta_2 \right) \wedge d\phi_1 \wedge d\phi_2 \quad (2.39) \\
H_3 &= 6g_s \mathbf{A}_4 M \left(1 + \frac{9g_s N_f}{4\pi} \log r + \frac{g_s N_f}{2\pi} \log \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2}\right) \frac{dr}{r} \\
&\quad \wedge \frac{1}{2} \left(\sin \theta_1 d\theta_1 \wedge d\phi_1 - \mathbf{B}_4 \sin \theta_2 d\theta_2 \wedge d\phi_2 \right) + \frac{3g_s^2 M N_f}{8\pi} \mathbf{A}_5 \left(\frac{dr}{r} \wedge e_\psi - \frac{1}{2} de_\psi \right) \\
&\quad \wedge \left(\cot \frac{\theta_2}{2} d\theta_2 - \mathbf{B}_5 \cot \frac{\theta_1}{2} d\theta_1 \right)
\end{aligned}$$

where we see that the background is exactly of the form presented in [38] except that there are asymmetry factors $\mathbf{A}_i, \mathbf{B}_i$. These asymmetry factors contain all the informations of the black hole etc in our background³. To order $\mathcal{O}(g_s N_f)$ these asymmetry factors are given by:

$$\begin{aligned}
\mathbf{A}_1 &= 1 + \frac{9g_s N_f}{4\pi} \cdot \frac{a^2}{r^2} \cdot (2 - 3 \log r) + \mathcal{O}(a^2 g_s^2 N_f^2) \\
\mathbf{B}_2 &= 1 + \frac{36a^2 \log r}{r^3 + 18a^2 r \log r} + \mathcal{O}(a^2 g_s^2 N_f^2) \\
\mathbf{A}_2 &= 1 + \frac{18a^2}{r^2} \cdot \log r + \mathcal{O}(a^2 g_s^2 N_f^2) \\
\mathbf{B}_1 &= 1 + \frac{81}{2} \cdot \frac{g_s N_f a^2 \log r}{4\pi r^2 + 9g_s N_f a^2 (2 - 3 \log r)} + \mathcal{O}(a^2 g_s^2 N_f^2) \\
\mathbf{A}_3 &= 1 - \frac{18a^2}{r^2} \cdot \log r + \mathcal{O}(a^2 g_s^2 N_f^2) \\
\mathbf{B}_3 &= 1 + \frac{36a^2 \log r}{r^2 - 18a^2 \log r} + \mathcal{O}(a^2 g_s^2 N_f^2) \\
\mathbf{A}_4 &= 1 - \frac{3a^2}{r^2} + \mathcal{O}(a^2 g_s^2 N_f^2), \quad \mathbf{B}_4 = 1 + \frac{3g_s a^2}{r^2 - 3a^2} + \mathcal{O}(a^2 g_s^2 N_f^2)
\end{aligned} \quad (2.40)$$

³One can easily see from these asymmetry factors that one of the two spheres is squashed. This squashing factor is of order $\mathcal{O}(g_s N_f)$ and therefore could have a perturbative expansion. Note also that although the resolution factor in the metric is hidden behind the horizon of the black hole the effect of this shows up in the fluxes. As far as we know, these details were first considered in [57]

$$\mathbf{A}_5 = 1 + \frac{36a^2 \log r}{r} + \mathcal{O}(a^2 g_s^2 N_f^2), \quad \mathbf{B}_5 = 1 + \frac{72a^2 \log r}{r + 36a^2 \log r} + \mathcal{O}(a^2 g_s^2 N_f^2)$$

These asymmetry factors tell us that corrections to the Ouyang background [38] come from $\mathcal{O}(a^2/r^2)$ onwards. Thus to complete the picture all we now need are the values for the axio-dilaton τ and the five form F_5 . If $z_1 = \mu$ gives the location of a single D7 brane as in (2.16), then from F theory one obtains $\tau \sim \log(z_1 - \mu)$ [38][58] near the D7 brane. In section 2.4 we will discuss in some detail how one obtains this form of τ . As $\tau = C_0 + ie^{-\phi}$ we get the following form for the dilaton and axion:

$$\begin{aligned} e^{-\phi} &= \frac{1}{g_s} - \frac{3N_f}{4\pi} \log r - \frac{N_f}{2\pi} \log \left(\sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right) + \mathcal{O}(\mu, a) \\ C_0 &= \frac{N_f}{4\pi} (\psi - \phi_1 - \phi_2) + \mathcal{O}(\mu, a) \\ F_5 &= \frac{1}{g_s} [d^4 x \wedge dh^{-1} + *(d^4 x \wedge dh^{-1})] \end{aligned} \quad (2.41)$$

where $\mathcal{O}(\mu, a)$ denotes all orders in μ and the resolution parameter a , h is the ten dimensional warp factor discussed above. Thus combining (2.96) and (2.41) our background can be written almost like the Ouyang background [38] with deviations given by (2.40).

In order to extract temperature from the geometry, we look at the metric in (2.23) in the near horizon limit $r \rightarrow r_h$. To be exact, we really need to start from the ten dimensional supergravity action and then integrate out the internal directions to obtain a five dimensional effective action. Minimization of that five dimensional effective action will give the five dimensional effective metric which is the same as integrating the five dimensional metric over the internal directions. That is the five dimensional metric is given by $g_{\mu\nu} = \int d\theta_i d\phi_i d\psi G_{\mu\nu}(\theta_i, \phi_i, \psi)$ where $G_{\mu\nu}$, $\mu, \nu = 0, \dots, 4$ is the metric as in (2.28) and we only integrate over the internal coordinates that $G_{\mu\nu}$ is a function of. We will denote the resulting warp factor for the five dimensional metric $g_{\mu\nu}$ as $h(r)$ for the ensuing analysis of temperature.

Now, looking at the r, t direction of the metric $g_{\mu\nu}$ and by change of variable,

under the assumption that $\int d\theta_k d\phi_j d\psi$ $g_i \approx g(r)$, $i, j, k = 1, 2$, we can define ρ^2 as:

$$\rho^2 = \frac{4\sqrt{h(r_h)g(r)}}{[g'(r_h)]^2} \quad (2.42)$$

so that the near horizon limit of five dimensional effective metric takes the following Rindler form:

$$ds^2 = -\rho^2 \frac{g'(r_h)^2}{4h(r_h)g(r_c)} dt_c^2 + d\rho^2 \quad (2.43)$$

where prime denotes differentiation with respect to r and we only wrote the r, t part of the metric in terms of new variable ρ and $t_c \equiv \sqrt{g(r_c)}t$. The reason behind rescaling time at fixed r_c is that with this time coordinate t_c , the five dimensional metric induces a four dimensional Minkowski metric at every r_c .

Now the temperature observed by the field theory with time coordinate t_c can be extracted by writing the metric in (2.43) in the following form

$$ds^2 = -4\pi^2 T_c^2 \rho^2 dt_c^2 + d\rho^2 \quad (2.44)$$

Thus comparing (2.43) and (2.44), we obtain the temperature T_c as:

$$T_c = \frac{g'(r_h)}{4\pi\sqrt{h(r_h)g(r_c)}} \quad (2.45)$$

In the limit where we have $g_1 = g_2 = g = 1 - \frac{r_h^4}{r^4}$, we can easily compute the corresponding temperature using the above formula (2.45). This is given by:

$$T_c = \frac{r_h}{\pi L^2} + \frac{r_h^5}{2\pi L^2 r_c^4} + \sum_{m,n,p} c_{mnp} \frac{r_h^m \log^n r_h}{r_c^p} \equiv T_b + \mathcal{O}(1/r_c) \quad (2.46)$$

where L is defined earlier, c_{mnp} is in general functions of (g_s, M, N, N_f) , and $T_b > T_{\text{deconf}}$ (where T_{deconf} is the deconfinement temperature) is the temperature at $r_c \rightarrow \infty$ i.e

$$T_b \equiv T_{\text{boundary}} = \frac{g'(r_h)}{4\pi\sqrt{h(r_h)}}, \quad r_h \equiv F(T_b) \equiv \mathcal{T} \quad (2.47)$$

where $F(T_b)$ can be obtained by inverting the first equation above. We can do this exactly once we know the black hole factor $g(r_h)$ as well as the warp factor $h(r_h)$ to

all orders in $g_s N_f, g_s M$. Here we identify the black hole horizon radius r_h with what characteristic temperature \mathcal{T} which is the only scale in the theory when there are no D7 branes.

With the knowledge of the dual gravity for the branes setups of KS and OKS models with temperature, one may ask whether Seiberg duality cascade can be realized from supergravity. The answer lies in identifying the gauge theory effective degrees of freedom using gravity. For AdS/CFT correspondence this identification is rather simple; the AdS throat radius L is directly related to the number of colors N of the gauge theory by $N = \frac{L^4}{4\pi g_s \alpha'^2}$ as can be seen from (2.2). Generalizing this result for the non-AdS/non-CFT models, one can immediately identify the effective degrees of freedom of the gauge theory with the throat radius $\tilde{L}^4(r) \equiv r^4 h$ where h is the warp factor appearing in (2.23). This gives

$$N_{\text{eff}}(r) = \frac{\tilde{L}^4(r)}{V g_s \alpha'^2} = \frac{r^4 h}{V g_s \alpha'^2} \quad (2.48)$$

where V is a constant which depends on the five dimensional compact manifold \mathcal{M}_5 . For example, if $\mathcal{M}_5 = T^{1,1}$, then $V = \frac{27}{4}\pi$ [33].

Now with the warp factor h given by (2.31), we see that N_{eff} grows with r and this gives a flow of effective degrees of freedom with changing r of the dual gravity. More precisely one can introduce a cutoff r_c in the geometry and evaluate the four dimensional dual field theory on the boundary $r = r_c$. The bulk geometry obtained this way extends from r_h to r_c and the dual gauge theory has degrees of freedom $N_{\text{eff}}(r_c)$. Note that the radial coordinate of dual gravity is identified as the energy scale of the gauge theory, that is $r = \Lambda$ and the geometry with cutoff r_c is dual to effective field theory at scale Λ_c . Of course one has to be specially careful in obtaining effective gauge theory Lagrangian from dual gravity. In particular one has to attach geometries from $r = r_c$ to $r = \infty$ in order to account for appropriate irrelevant and marginal operators that have been integrated out. This procedure of attaching UV caps to geometries will be discussed in detail in the coming sections.

As N_{eff} grows with r , we conclude from gravity that effective degrees of freedom

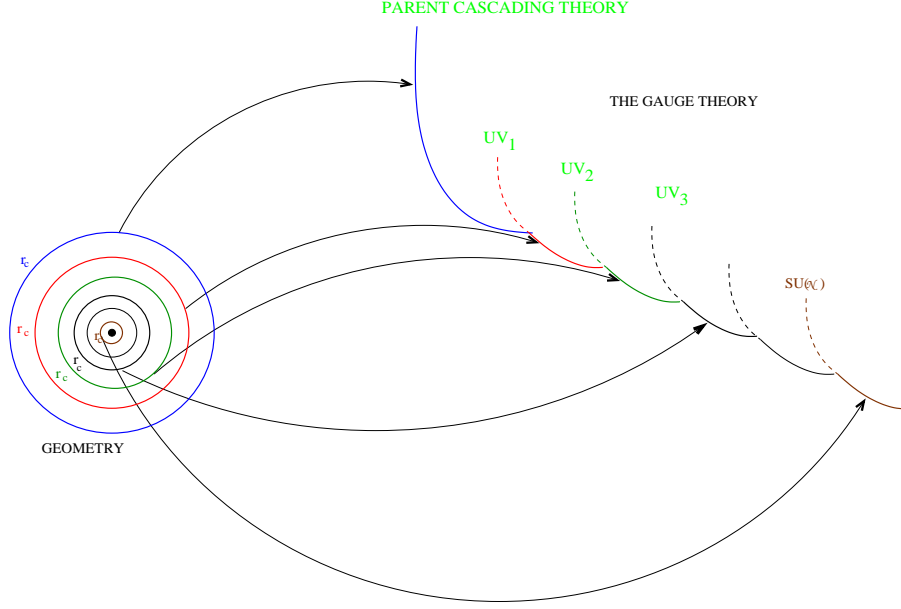


Figure 2.13: Seiberg duality cascade captured by dual geometry.

grow in the UV and shrink in the IR, which is exactly consistent with Seiberg duality cascade. At some UV scale Λ_{UV} , we start with $SU(N + M) \times SU(N)$ and as we go to lower and lower energy scales, we can describe the theory with lower and lower rank gauge groups using a cascade of Seiberg dualities. Thus supergravity captures the generic feature of Seiberg duality cascade and N_{eff} at some value of r , that is at some value of scale Λ describes the overall degrees of freedom of the gauge theory. Fig 2.13 shows a simplified depiction of how an effective field theory at scale Λ_c can be mapped to a geometry with cutoff r_c . For more details consult [46, 57].

Although generic features of the duality cascade is captured by gravity, there are crucial distinctions between the cascade and the flow obtained from dual gravity. To describe a theory with its Seiberg dual, a key requirement is that the coupling has to be either very strong or very weak. In the space of coupling constants, one must be at the boundary curve of Fig 2.14. For $N, M \sim \mathcal{O}(1)$, the t'Hooft couplings are roughly equal to the gauge couplings, that is $\lambda_k \sim g_k$ so we can treat g_k as the t'Hooft couplings. Then Seiberg duality can be performed at a scale when t'Hooft coupling λ_k of $SU(N + M)$ is very large and λ_{k-1} of $SU(N)$ is very small. On the other hand

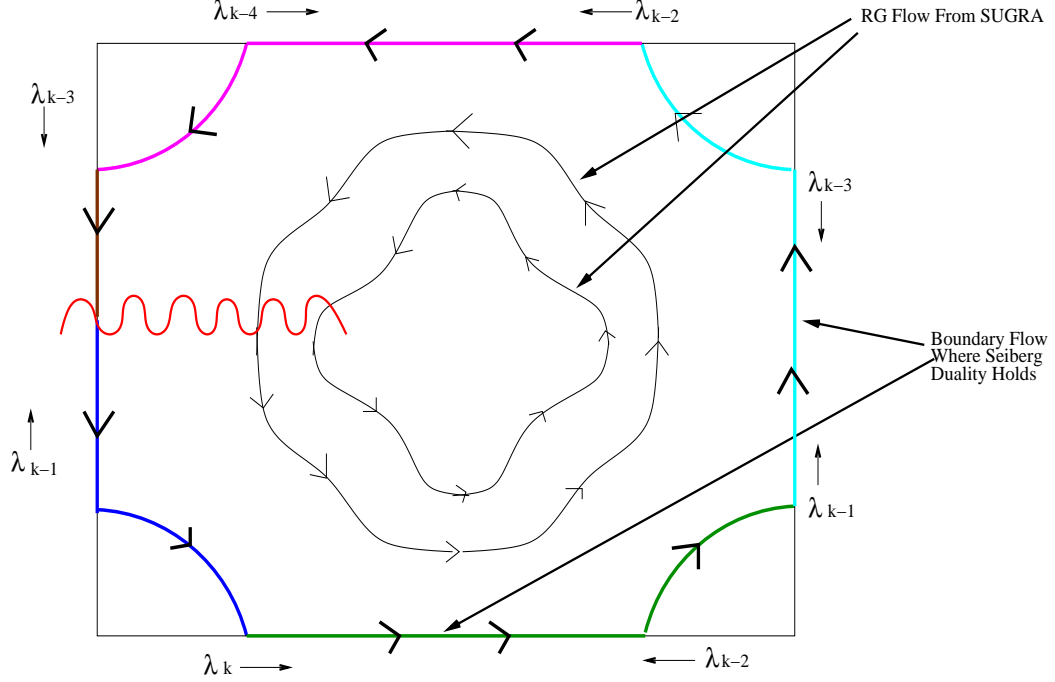


Figure 2.14: Effective field theories at various scales along with their dual geometry with cutoffs.

supergravity solution is valid when $N, M \gg 1$ and using (2.21) one obtains that both t'Hooft couplings λ_k, λ_{k-1} are very large. This means the SUGRA solution lies in the interior of coupling constant surface of Fig 2.14 and not on the boundary curve where Seiberg duality holds. The situation is depicted in Fig 2.14.

The regime where supergravity solution holds is precisely beyond the phase space where Seiberg duality holds. The flow captured by supergravity is a smooth RG flow with continuously changing degrees of freedom rather than a step by step reduction of gauge group by some units (for KS gauge theory by M units) as described by the cascade. However supergravity may capture crucial features of the gauge theory which are independent of smooth RG flow or patchy Seiberg duality cascade.

With a clear understanding of the gauge theories arising from the branes and their dual gravity, we would like to know how to compute relevant gauge theory observables. In the following section we discuss the precise mapping between gauge theory correlation functions and geometry.

2.3 Gauge Theory Observables from Dual Gravity

In order to compute expectation values of certain observables of the gauge theory, one first computes the partition function. For a system at thermal equilibrium, various thermodynamic quantities such as free energy, pressure, entropy etc. can easily be obtained from the partition function. On the other hand propagators of quantum fields and subsequently scattering amplitudes of particles can be easily evaluated by taking the functional derivative of the partition function with respect to some appropriate sources. Thus once we know the partition function of a quantum field theory, we can essentially determine its thermodynamic properties.

As the Hilbert space of certain quantum field theories arising from brane excitations is contained in the Hilbert space of the geometry sourced by the branes, there should be a one-to-one correspondence between the partition functions of the gauge theory and the dual gravity. A precise mapping was proposed by Witten [59] and subsequently by the authors of [60]. Witten's proposal states that the partition function of the strongly coupled quantum field theory should be identified with the partition function of the weakly coupled classical gravity. In particular if we are interested in computing the expectation value of an operator $\langle \mathcal{O} \rangle$ with a source ϕ_0 , one can make the following identification of the partition function as a functional of the source ϕ_0 :

$$\begin{aligned} \mathcal{Z}_{\text{gauge}}[\phi_0] &\equiv \langle \exp \int_{M^4} \phi_0 \mathcal{O} \rangle = \mathcal{Z}_{\text{gravity}}[\phi_0] \\ &\equiv \exp(S_{\text{SUGRA}}[\phi_0] + S_{\text{GH}}[\phi_0] + S_{\text{counterterm}}[\phi_0]) \end{aligned} \quad (2.49)$$

where M^4 is a Minkowski manifold, S_{GH} is the Gibbons-Hawking boundary term [61], ϕ_0 should be understood as a fluctuation over a given configuration of field and $S_{\text{counterterm}}$ is the counter-term action added to renormalize the action. We will briefly describe how each term is obtained from supergravity action and their necessity.

First observe that ϕ is a bulk field, which is in general a function of the coordinates of the geometry and ϕ_0 is the boundary value of ϕ . One integrates the classical ten dimensional gravity action (2.19) over the compact manifold \mathcal{M}_5 to obtain an effective action S_5^{eff} for a five dimensional manifold. For AdS/CFT correspondence

the five dimensional manifold one obtains after integration over $\mathcal{M}_5 = S_5$ is of course AdS_5 geometry whereas for non-Ads geometries described in the previous section, one obtains modified AdS space. The bulk five dimensional geometry has a boundary at $r = \infty$ and the boundary describes a four dimensional manifold with an induced Minkowski flat metric. Thus the boundary value of ϕ

$$\phi_0(t, x, y, z) \equiv \phi(r = \infty, t, x, y, z) \quad (2.50)$$

is a field which lives in four dimensional flat space and plays the role of source in the four dimensional gauge theory.

Now to obtain the gravity action as a functional of ϕ_0 , one integrates the five dimensional effective action S_5^{eff} over the radial coordinate r and we are left with gravity action only being function of ϕ_0 . Given a boundary value ϕ_0 , one can uniquely build the bulk field ϕ and write the bulk action for ϕ as a function of the boundary value ϕ_0 . This was shown first by Witten [59] for AdS bulk fields and for the geometries in [57, 62] which are perturbations on AdS space, the same argument holds.

It turns out that in general for the gravity action $S[\phi]$ to be stationary under perturbation $\phi \rightarrow \phi + \delta\phi$, one needs to add surface terms which are known as Gibbons-Hawking terms [61]. On the other hand, after the radial integral is done, for some sources ϕ , $S[\phi_0]$ becomes infinite and one needs to regularize the action to obtain finite expectation values for operators. The regularization requires addition of extra terms which cancel the infinities that appear in the gravity action and are denoted by $S_{\text{counterterm}}$. Keeping all this in mind, one takes the functional derivative of (2.49) with respect to ϕ_0 to obtain

$$\begin{aligned} \langle \mathcal{O} \rangle &= \frac{\delta \mathcal{Z}_{\text{gauge}}}{\delta \phi_0} = \frac{\delta \mathcal{Z}_{\text{gravity}}}{\delta \phi_0} \\ &\sim \exp(S_{\text{total}}[\phi_0]) \frac{\delta S_{\text{total}}[\phi_0]}{\delta \phi_0} \Big|_{\phi_0=0} \end{aligned} \quad (2.51)$$

where $S_{\text{total}}[\phi_0] = S_{\text{SUGRA}}[\phi_0] + S_{\text{GH}}[\phi_0] + S_{\text{counterterm}}[\phi_0]$.

Observe that in the usual AdS/CFT case we consider the action at the boundary to map it directly to the dual gauge theory side. For general gauge/gravity duali-

ties, there are many possibilities of defining different gauge theories at the boundary depending on how we cut-off the geometry and add UV caps. We will elaborate the addition of UV caps in section **2.3.2** and in the following section **2.3.1**, we briefly describe renormalization of the supergravity action.

2.3.1 Holographic Renormalization

As already mentioned, supergravity action $S_{\text{sugra}}[\phi_0]$, for some sources ϕ_0 , becomes infinity and one needs to regularize the action. In particular if we want to compute the stress tensor T^{pq} , $p, q = 0, 1, 2, 3$ of a field theory in four dimensional flat space time, then the source ϕ_0 is the metric η_{pq} , i.e the Minkowski metric. It turns out that if we naively integrate over the radial coordinate r of S_5^{eff} , the resulting action as a function of η_{pq} blows up.

First observe that it is indeed possible to write the five dimensional supergravity action for AdS or modified AdS space as a functional of flat four dimensional metric. This is because for both AdS and non-AdS space (KS and OKS background with black hole), the five dimensional metric has the following form:

$$\begin{aligned} ds^2 &= -\frac{g_1}{\sqrt{h}}dt^2 + \frac{1}{\sqrt{h}}(dx^2 + dy^2 + dz^2) + \frac{\sqrt{h}}{g_2}dr^2 \\ &\equiv g_{\mu\nu}dx^\mu dx^\nu \end{aligned} \quad (2.52)$$

At the boundary $r = \infty$, $g_1 \rightarrow 1$, and the induced metric is $\eta_{pq}/\sqrt{h(\infty)}$ which is a constant factor times the Minkowski metric. Thus $S_5^{\text{eff}}[g_{\mu\nu}]$ can be written as a function of η_{pq} once the radial integral is done.

Now for AdS space, starting with (2.19) with only $\tilde{F}_5 \neq 0$ being the source, integrating over the internal coordinates give the following action:

$$S_{AdS_5}^{\text{eff}} = \int d^4x \int_{r_h}^{r_c} dr \sqrt{g} (R_5 + \Lambda) \quad (2.53)$$

where $g = \det g_{\mu\nu}$, Λ is the cosmological constant and $r_c \rightarrow \infty$. Note that the above action is not only the action for AdS space but also for asymptotic AdS spaces i.e. geometries $G_{\mu\nu}$ such that $\lim_{r \rightarrow \infty} G_{\mu\nu} = g_{\mu\nu}$ with $g_{\mu\nu}$ given by (2.52) where $h = L^4/r^4$ [63].

After the r integral is done, the result contains terms which diverge as $r_c \rightarrow \infty$. As the AdS or asymptotic AdS metric is a power series in r , the divergent terms are precisely the ones containing positive powers of r_c and by subtracting these terms we get a finite result for the action. This subtraction procedure can be done by introducing counter term S_{counter} to the action (2.53) such that they cancel the infinities coming from it. The renormalization scheme mentioned here was first proposed by Skenderis and for more details please consult [63]-[67].

With a finite renormalized action $S_{\text{AdS}_5}^{\text{ren}}[\eta_{pq}]$, and using the equation of motion for the metric $g_{\mu\nu}$, one can take the derivative with respect to η_{pq} to obtain the following result for the stress tensor [65]

$$T^{pq} = \frac{1}{4\pi G_N} \left(g_{(4)pq} - \frac{1}{8} g_{(0)pq} \left[(\text{Tr} g_{(2)})^2 - \text{Tr} g_{(2)^2} \right] - \frac{1}{2} (g_{(2)}^2)_{pq} + \frac{1}{4} g_{(2)pq} \text{Tr} g_{(2)} \right) \quad (2.54)$$

where G_N is the five dimensional Newton's constant and $g_{(l)}$ is defined through the bulk metric $G_{\mu\nu}$ the following way:

$$\begin{aligned} G_{\mu\nu} dx^\mu dx^\nu &= \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} \tilde{g}_{pq}(x^p, \rho) dx^p dx^q \\ \tilde{g}(x, \rho) &= g_{(0)}(x) + \dots + \rho^{d/2} g_{(d)} + \dots \end{aligned} \quad (2.55)$$

where $\rho = 1/r^2$. Thus $g_{(l)}$ are Taylor coefficients in the expansion of the four dimensional metric $\tilde{g}_{pq}(x^p, \rho)$ which induces the four dimensional boundary metric $g_{(0)pq}$ at the boundary $\rho = 0$.

The story gets somewhat more involved for non-AdS geometries and we will briefly review the key points of holographic renormalization of dual supergravity for non conformal field theories. Details of the analysis can be found in our work [57, 62].

The divergent terms appearing in the supergravity action are uniquely determined once we know the form of the warp factor h in (2.23). Observe that the logarithms appearing in h (2.31) results from the logarithmic running of the dilaton field (2.41) and that of NS-NS field B_2 . The dilaton only behaves logarithmically near the location of the D7 brane and its global behavior is determined by F theory. For large r and away from any D7 branes, we expect the dilaton to behave as a constant. In fact in section 2.4 we will analyze the precise running of the axio-dilaton field for all r where

we will also see how the logarithmic running of NS-NS B_2 field can be modified to give finite value for B_2 at large r . With B_2 and axio-dilaton τ behaving as logarithms in the local neighborhood of some r but approaching constant finite values for large r , in [57] we proposed the following form of the warp factor:

$$h = \frac{L^4}{r^{4-\epsilon_1}} + \frac{L^4}{r^{4-2\epsilon_2}} - \frac{2L^4}{r^{4-\epsilon_2}} + \frac{L^4}{r^{4-r\epsilon_2^2/2}} \equiv \sum_{\alpha=1}^4 \frac{L_{(\alpha)}^4}{r_{(\alpha)}^4} \quad (2.56)$$

where $\epsilon_i, r_{(\alpha)}$ etc are defined as:

$$\begin{aligned} \epsilon_1 &= \frac{3g_s M^2}{2\pi N} + \frac{g_s^2 M^2 N_f}{8\pi^2 N} + \frac{3g_s^2 M^2 N_f}{8\pi N} \log \left(\sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right), \quad \epsilon_2 = \frac{g_s M}{\pi} \sqrt{\frac{2N_f}{N}} \\ r_{(\alpha)} &= r^{1-\epsilon_{(\alpha)}}, \quad \epsilon_{(1)} = \frac{\epsilon_1}{4}, \quad \epsilon_{(2)} = \frac{\epsilon_2}{2}, \quad \epsilon_{(3)} = \frac{\epsilon_2}{4}, \quad \epsilon_{(4)} = \frac{\epsilon_2^2}{8} \\ r_{(\pm\alpha)} &= r^{1\mp\epsilon_{(\alpha)}}, \quad L_{(1)} = L_{(2)} = L_{(4)} = L^4, \quad L_{(3)} = -2L^4 \end{aligned} \quad (2.57)$$

which makes sense because we can make ϵ_i to be very small. Note that the choice of ϵ_i doesn't require us to have $g_s N_f$ small (although we consider it here). In fact we *can* have all (N, M, N_f) large but ϵ_i small. A simple way to achieve this would be to have the following scaling behaviors of (g_s, N, M, N_f) :

$$g_s \rightarrow \epsilon^\alpha, \quad M \rightarrow \epsilon^{-\beta}, \quad N_f \rightarrow \epsilon^{-\kappa}, \quad N \rightarrow \epsilon^{-\gamma} \quad (2.58)$$

where $\epsilon \rightarrow 0$ is the tunable parameter. Therefore all we require to achieve that is to allow:

$$\alpha + \gamma > 2\beta + \kappa, \quad \alpha > \kappa, \quad \gamma > \alpha \quad (2.59)$$

where the last inequality can keep $g_s N_f$ small. Thus $g_s N, g_s M$ are very large, but $g_s, \frac{g_s M^2}{N}, g_s N_f$ are all very small to justify our expansions (and the choice of supergravity background)⁴.

⁴For example we can have g_s going to zero as $g_s \rightarrow \epsilon^{5/2}$ and (N, M, N_f) going to infinities as $(\epsilon^{-8}, \epsilon^{-3}, \epsilon^{-1})$ respectively. This means $(g_s N, g_s M)$ go to infinities as $(\epsilon^{-11/2}, \epsilon^{-1/2})$ respectively, and $(g_s N_f, g_s^2 M N_f, g_s M^2/N)$ go to zero as $(\epsilon^{3/2}, \epsilon, \epsilon^{9/2})$ respectively. This is one limit where we can have well defined UV completed gauge theories. Note however that for the kind of background that we have been studying one cannot make N_f large because of the underlying F-theory constraints [81][58][82]. Since we only require $g_s N_f$ small, large or small N_f choices do not change any of our results.

The warp factor (2.56) has a good behavior at infinity and reproduces the $\mathcal{O}(g_s N_f)$ result locally. Our conjecture then would be the complete form of the warp factor at large r will be given by sum over α as in (2.57) but now α can take values $1 \leq \alpha \leq \infty$. This conjecture in fact justifies the holographic renormalizability of our boundary theory as we will find out shortly.

With the warp factor given by (2.56), we introduce perturbation $l_{\mu\nu}$ to background five dimensional metric $g_{\mu\nu}$ given by (2.52). Recall that for non-AdS space the five dimensional effective action S_5^{eff} derived from ten dimensional action (2.19) takes the form

$$S_5^{\text{eff}} = \int d^5x \sqrt{G} (R + \Lambda(G_{\mu\nu})) \quad (2.60)$$

where $G_{\mu\nu} = g_{\mu\nu} + l_{\mu\nu}$ and the source $\Lambda(G_{\mu\nu})$ is in general a function of the metric. As $l_{\mu\nu}$ is a perturbation, one can write the action (2.60) as a Taylor series in the perturbation. Considering terms only up to quadratic order in $l_{\mu\nu}$ one obtains:

$$\begin{aligned} \mathcal{S}^{(1)}[\Phi] = & \int \frac{d^4q}{(2\pi)^4 \sqrt{g(r_c)}} \int dr \left\{ \frac{1}{2} A_1^{mn}(r, q) [\Phi_m^{[1]}(r, q) \Phi_n'^{[1]}(r, -q) + \Phi_m''^{[1]}(r, q) \Phi_n^{[1]}(r, -q)] \right. \\ & + B_1^{mn}(r, q) \Phi_m'^{[1]}(r, q) \Phi_n^{[1]}(r, -q) + \frac{1}{2} C_1^{mn}(r, q) [\Phi_m'^{[1]}(r, q) \Phi_n^{[1]}(r, -q) + \Phi_m^{[1]}(r, q) \Phi_n'^{[1]}(r, -q)] \\ & \left. + D_1^{mn}(r, q) \Phi_m^{[1]}(r, q) \Phi_n^{[1]}(r, -q) + \mathcal{T}_1^m(r, q) \Phi_m^{[1]}(r, q) + E_1^m \Phi_m'^{[1]}(r, q) + F_1^m \Phi_m''^{[1]}(r, q) \right\} \end{aligned} \quad (2.61)$$

where $m, n = 1, \dots, 5$, prime denotes differentiation with respect to r , the script [1] denote the *total* background to $\mathcal{O}(g_s N_f, g_s M^2/N)$; and the explicit expressions for $A_1^{mn}, B_1^{mn}, C_1^{mn}, F_1^m$ for a specific case are given in Appendix D of [57]. We have also defined $\Phi_m^{[1]}(r, q)$ in the following way (with $q_0 \equiv \omega \sqrt{g(r_c)}$ as before):

$$\Phi_m^{[1]}(r, q) = \int \frac{d^4x}{(2\pi)^4 \sqrt{g(r_c)}} e^{i(q_0 t - q_1 x - q_2 y - q_3 z)} l_{mm}(t, r, x, y, z) \quad (2.62)$$

We will see that the effective four dimensional boundary action is independent of D_1^{mn} and \mathcal{T}_1^m and hence their explicit expressions do not appear in the appendix of [57]. Furthermore, note that the derivative terms in (2.61) all come exclusively from

$\sqrt{-G}R$, whereas fluxes contribute powers of $\Phi^{[1]}$ but no derivative interactions. In the following we keep up to quadratic orders, and therefore the contributions from the fluxes will appear in D_1^{mn} and E_1^m .

The equation of motion for $\Phi_n^{[1]}(r, -q)$ is given by:

$$\begin{aligned} \frac{1}{2} [A_1^{mn}(r, q) \Phi_n^{[1]}(r, -q)]'' - [B_1^{mn}(r, q) \Phi_n'^{[1]}(r, -q)]' - \frac{1}{2} [C_1^{mn}(r, q) \Phi_n^{[1]}(r, -q)]' \\ + D_1^{mn}(r, q) \Phi_n^{[1]}(r, -q) + \frac{1}{2} A_1^{mn}(r, q) \Phi_n''^{[1]}(r, -q) - \frac{1}{2} C_1^{mn}(r, q) \Phi_n'^{[1]}(r, -q) \\ + \mathcal{T}_1^m(r, q) - E_1^m(r, q) + F_1'^m(r, q) = 0 \end{aligned} \quad (2.63)$$

The next few steps are rather standard and so we will quote the results. The variation of the action (2.61) can be written in terms of the variations $\delta\Phi_m^{[1]}(r, q)$ and $\delta\Phi_n^{[1]}(r, -q)$ in the following way⁵:

$$\begin{aligned} \delta\mathcal{S}^{(1)} = \frac{1}{2} \int \frac{d^4q}{(2\pi)^4 \sqrt{g(r_c)}} \int_{r_h}^{r_c} dr \left\{ [(A_1^{mn} \Phi_m^{[1]})'' - (2B_1^{mn} \Phi_m'^{[1]})' + C_1^{mn} \Phi_m'^{[1]} + 2D_1^{mn} \Phi_m^{[1]} \right. \\ + A_1^{mn} \Phi_m''^{[1]} - (C_1^{mn} \Phi_m^{[1]})'] \delta\Phi_n^{[1]} + [(A_1^{mn} \Phi_n^{[1]})'' - (2B_1^{mn} \Phi_n'^{[1]})' + C_1^{mn} \Phi_n'^{[1]} + 2D_1^{mn} \Phi_n^{[1]} \\ + A_1^{mn} \Phi_n''^{[1]} - (C_1^{mn} \Phi_n^{[1]})'] \delta\Phi_m^{[1]} + 2(\mathcal{T}_1^m - E_1^m + F_1'^m) \delta\Phi_m^{[1]} \\ \partial_r [A_1^{mn} \Phi_m^{[1]} \delta\Phi_n^{[1]} - (A_1^{mn} \Phi_m^{[1]})' \delta\Phi_n^{[1]} + 2B_1^{mn} \Phi_m'^{[1]} \delta\Phi_n^{[1]} + C_1^{mn} \Phi_m \delta\Phi_n^{[1]} + 2B_1^{mn} \Phi_n'^{[1]} \delta\Phi_m^{[1]} \\ \left. + C_1^{mn} \Phi_n^{[1]} \delta\Phi_m^{[1]} + 2E_1^m \delta\Phi_m^{[1]} + 2F_1^m \delta\Phi_m'^{[1]} - 2F_1'^m \delta\Phi_m^{[1]} + A_1^{mn} \Phi_n^{[1]} \delta\Phi_m'^{[1]} - (A_1^{mn} \Phi_n^{[1]})' \delta\Phi_m^{[1]}] \right\} \end{aligned} \quad (2.64)$$

which includes the equations of motion as well as the boundary term. We can then write the variation of the action $\delta\mathcal{S}^{(1)}$ in the following way:

$$\begin{aligned} \delta\mathcal{S}^{(1)} = \int \frac{d^4q}{(2\pi)^4 \sqrt{g(r_c)}} \left\{ \int_{r_h}^{r_c} dr \left[(\text{EOM for } \Phi_n^{[1]}) \delta\Phi_m^{[1]} + (\text{EOM for } \Phi_m^{[1]}) \delta\Phi_n^{[1]} \right] \right. \\ + \frac{1}{2} [(2B_1^{mn} - A_1^{mn})(\Phi_m'^{[1]} \delta\Phi_n^{[1]} + \Phi_n'^{[1]} \delta\Phi_m^{[1]}) + (C_1^{mn} - A_1'^{mn})(\Phi_m^{[1]} \delta\Phi_n^{[1]} + \Phi_n^{[1]} \delta\Phi_m^{[1]}) \\ \left. + 2(E_1^m - F_1'^m) \delta\Phi_m^{[1]} + A_1^{mn} \Phi_m^{[1]} \delta\Phi_n'^{[1]} + A_1^{mn} \Phi_n^{[1]} \delta\Phi_m'^{[1]} + 2F_1^m \delta\Phi_m'^{[1]}]_{\text{boundary}} \right\} \end{aligned} \quad (2.65)$$

⁵Henceforth, unless mentioned otherwise, $\Phi_m^{[1]}, \Phi_n^{[1]}$ will always mean $\Phi_m^{[1]}(r, q)$ and $\Phi_n^{[1]}(r, -q)$ respectively. Similar definitions go for the variations $\delta\Phi_m^{[1]}$ and $\delta\Phi_n^{[1]}$.

where by an abuse of notation by the “boundary” here, and the next couple of pages (unless mentioned otherwise), we mean that the functions are all measured at r_h and r_c i.e the horizon and the cut-off respectively⁶. It is now easy to see why D_1^{mn} , \mathcal{T}_1^{mn} and E_0^{mn} etc do not appear in the boundary action. Finally, we need to add another boundary term to (2.65) to cancel of the term proportional to $\delta\Phi'_n$. This is precisely the Gibbons-Hawking term [61]:

$$\mathcal{K}_1 = -\frac{1}{2} \int \frac{d^4 q}{(2\pi)^4 \sqrt{g(r_c)}} \left(A_1^{mn} \Phi_m^{[1]} \Phi_n'^{[1]} + A_1^{mn} \Phi_n^{[1]} \Phi_m'^{[1]} + 2F_1^n \Phi_n'^{[1]} \right) \Big|_{\text{boundary}} \quad (2.66)$$

Taking the variation of (2.66) $\delta\mathcal{K}_1$ we get terms proportional to $\delta\Phi'^{[1]}$ as well as $\delta\Phi^{[1]}$. Adding $\delta\mathcal{K}_1$ to $\delta\mathcal{S}^{(1)}$ we can get rid of all the $\delta\Phi'^{[1]}$ terms from (2.65). This means we can alternately state that the boundary theory should have the following constraints⁷:

$$\delta\Phi_m'^{[1]}(r_c, q) = \delta\Phi_n'^{[1]}(r_c, -q) = 0 \quad (2.67)$$

With all the above considerations we can present our final result for the boundary action. Putting the equations of motion constraints on (2.65), as well as the derivative constraints (2.67), we can show that the variation (2.65) can come from the following boundary 3 + 1 dimensional action:

$$\begin{aligned} \mathcal{S}^{(1)} = & \int \frac{d^4 q}{(2\pi)^4 \sqrt{g(r_{\max})}} \left\{ [C_1^{mn}(r, q) - A_1'^{mn}(r, q)] \Phi_m^{[1]}(r, q) \Phi_n^{[1]}(r, -q) \right. \\ & + [B_1^{mn}(r, q) - A_1^{mn}(r, q)] [\Phi_m'^{[1]}(r, q) \Phi_n^{[1]}(r, -q) + \Phi_m^{[1]}(r, q) \Phi_n'^{[1]}(r, -q)] \\ & \left. + (E_1^m - F_1'^m) \Phi_m^{[1]}(r, q) \right\} \Big|_{r_h}^{r_c} \quad (2.68) \end{aligned}$$

However the above action diverges, as one can easily check from the explicit expressions for A_1^{mn} , B_1^{mn} , C_1^{mn} , E_1^m and F_1^m for the specific case worked in Appendix D

⁶Note that we have not carefully described the degrees of freedom at the boundary as yet. For large enough r_c we expect large degrees of freedom at the UV. This would mean that the contributions to various gauge theories from these degrees of freedom would go like $e^{-\mathcal{N}_{\text{eff}}}$, which would be negligible. Thus unless we cut-off the geometry at $r = r_c$ and add UV caps with specified degrees of freedom we are in principle only describing the parent cascading theory. For this theory of course \mathcal{N}_{eff} is infinite at the boundary, which amounts to saying that UV degrees of freedom don't contribute anything here. We will, however, give a more precise description a little later.

⁷One can impose similar constraints at the horizon also.

of [57]. Indeed, comparing it to the known AdS results, we observe that from the boundary there are terms proportional to r_c^4 in each of $C_1^{mn}, A_1'^{mn}, E_1^m$ and $F_1'^m$ and proportional to r_c^5 in A_1^{mn}, B_1^{mn} . As $r_c \rightarrow \infty$ the action diverges so as it stands r_c cannot completely specify the UV degrees of freedom at the scale Λ_c . Thus we need to regularize/renormalize it before taking functional derivative of it. This renormalization procedure will give us a finite boundary theory from which one could get meaningful results of the dual gauge theory.

Once we express the warp factors in terms of power series in $r_{(\alpha)}$ (2.56) the renormalizability procedure becomes much simpler. Note however that this renormalization is only in classical sense, as the procedure will involve removing the infinities in (2.68) by adding counter-terms to it. Comparing with the known AdS results, and the specific example presented in Appendix C of [57], one can argue that the infinities in (2.68) arise from the following three sources:

$$\begin{aligned}
1. \quad & C_1^{mn}(r_c, q) - A_1'^{mn}(r_c, q) = \sum_{\alpha} H_{|\alpha|}^{mn}(q) r_{c(\alpha)}^4 + \text{finite terms} \\
2. \quad & B_1^{mn}(r_c, q) - A_1^{mn}(r_c, q) = \sum_{\alpha} K_{|\alpha|}^{mn}(q) r_{c(\alpha)}^5 + \text{finite terms} \\
3. \quad & E_1^m(r_c, q) - F_1'^m(r_c, q) = \sum_{\alpha} I_{|\alpha|}^m(q) r_{c(\alpha)}^4 + \text{finite terms} \quad (2.69)
\end{aligned}$$

where in the above expressions we are keeping α arbitrary so that it can in general take both positive and negative values; and the finite terms above are of the form $r_{c(\alpha)}^{-n}$ with $n \geq 1$. Therefore to regularize, first we write the metric perturbation also as a series in $1/r_{(\alpha)}$:

$$\Phi_n^{[1]} = \sum_{k=0}^{\infty} \sum_{\alpha} \frac{s_{nn}^{(k)[\alpha]}}{r_{(\alpha)}^k} \quad (2.70)$$

where the above relation could be easily derived using (2.62), taking the background warp factor correctly.

Plugging in (2.70) and (2.69) in (2.68) we can easily extract the divergent parts of it. Thus the counter-terms are given by:

$$\begin{aligned}
\mathcal{S}_{\text{counter}}^{(1)} = & \int \frac{d^4 q}{2(2\pi)^4 \sqrt{g(r_{\text{max}})}} \sum_{\alpha, \beta, \gamma} \left\{ H_{|\alpha|}^{mn} \left[s_{mm}^{(0)[\beta]} s_{nn}^{(0)[\gamma]} r_{(\alpha)}^4 + \left(s_{mm}^{(1)[\beta]} s_{nn}^{(0)[\gamma]} r_{(\alpha_1)}^3 \right. \right. \right. \\
& \left. \left. + s_{mm}^{(0)[\beta]} s_{nn}^{(1)[\gamma]} r_{(\alpha_2)}^3 \right) + \left(s_{mm}^{(2)[\beta]} s_{nn}^{(0)[\gamma]} r_{(\alpha_3)}^2 + s_{mm}^{(0)[\beta]} s_{nn}^{(2)[\gamma]} r_{(\alpha_4)}^2 + s_{mm}^{(1)[\beta]} s_{nn}^{(1)[\gamma]} r_{(\alpha_5)}^2 \right) \right\}
\end{aligned}$$

$$\begin{aligned}
 & + \left(s_{mm}^{(3)[\beta]} s_{nn}^{(0)[\gamma]} r_{(\alpha_6)} + s_{mm}^{(0)[\beta]} s_{nn}^{(3)[\gamma]} r_{(\alpha_7)} + s_{mm}^{(2)[\beta]} s_{nn}^{(1)[\gamma]} r_{(\alpha_8)} + s_{mm}^{(1)[\beta]} s_{nn}^{(2)[\gamma]} r_{(\alpha_9)} \right) r_{(\alpha_9)} \Big] \\
 & + K_{|\alpha|}^{mn} \left[- \left(s_{mm}^{(0)[\beta]} s_{nn}^{(1)[\gamma]} r_{(\alpha_{10})}^3 + s_{mm}^{(1)[\beta]} s_{nn}^{(0)[\gamma]} r_{(\alpha_{11})}^3 \right) - \left(2 s_{mm}^{(1)[\beta]} s_{nn}^{(1)[\gamma]} r_{(\alpha_{12})}^2 \right. \right. \\
 & + 2 s_{mm}^{(0)[\beta]} s_{nn}^{(2)[\gamma]} r_{(\alpha_{13})}^2 + 2 s_{nn}^{(0)[\beta]} s_{mm}^{(2)[\gamma]} r_{(\alpha_{14})}^2 \Big) - \left(2 s_{mm}^{(1)[\beta]} s_{nn}^{(2)[\gamma]} r_{(\alpha_{15})} + s_{mm}^{(2)[\beta]} s_{nn}^{(1)[\gamma]} r_{(\alpha_{16})} \right. \\
 & + 3 s_{mm}^{(0)[\beta]} s_{nn}^{(3)[\gamma]} r_{(\alpha_{17})} + 2 s_{nn}^{(1)[\beta]} s_{mm}^{(2)[\gamma]} r_{(\alpha_{18})} + s_{nn}^{(2)[\beta]} s_{mm}^{(1)[\gamma]} r_{(\alpha_{19})} + 3 s_{nn}^{(0)[\beta]} s_{mm}^{(3)[\gamma]} r_{(\alpha_{20})} \Big) \Big] \\
 & + I_{|\alpha|}^m \theta(r_0 - r) \left(s_{mm}^{(0)[\beta]} r_{(\alpha)}^4 + s_{mm}^{(1)[\beta]} r_{(\alpha)}^3 + s_{mm}^{(2)[\beta]} r_{(\alpha)}^2 + s_{mm}^{(3)[\beta]} r_{(\alpha)} \right) \Big\} \quad (2.71)
 \end{aligned}$$

with an equal set of terms with $r_{(-\alpha_i)}$. In the above expression $r_{(\alpha_i)} \equiv r^{1-\epsilon_{(\alpha_i)}}$; and as before, the integrand is defined at the horizon r_h and the cutoff r_c . The other variables namely, $s_{mm}^{(k)[\beta]}$, $H_{|\alpha|}^{mn}$, $K_{|\alpha|}^{mn}$ and $I_{|\alpha|}^m$ are independent of r but functions of q^i . For one specific case their values are given in Appendix C of [57]. Finally the $\epsilon_{(\alpha_i)}$ can be defined by the following procedure. Lets start with the expression:

$$\begin{aligned}
 H_{|\alpha|}^{mn} s_{mm}^{(a)[\beta]} s_{nn}^{(b)[\gamma]} r_{(\alpha_k)}^p & \equiv H_{|\alpha|}^{mn} s_{mm}^{(a)[\beta]} s_{nn}^{(b)[\gamma]} \frac{r_{(\alpha)}^4}{r_{(\beta)}^a r_{(\gamma)}^b} \\
 K_{|\alpha|}^{mn} s_{mm}^{(c)[\beta]} s_{nn}^{(d)[\gamma]} r_{(\alpha_l)}^q & \equiv K_{|\alpha|}^{mn} s_{mm}^{(c)[\beta]} s_{nn}^{(d)[\gamma]} \frac{r_{(\alpha)}^5}{r_{(\beta)}^c r_{(\gamma)}^d} \quad (2.72)
 \end{aligned}$$

from where one can easily infer:

$$\begin{aligned}
 p &= 4 - a - b, & \epsilon_{(\alpha_k)} &= \frac{4\epsilon_{(\alpha)} - a\epsilon_{(\beta)} - b\epsilon_{(\gamma)}}{4 - a - b} \\
 q &= 5 - c - d, & \epsilon_{(\alpha_l)} &= \frac{5\epsilon_{(\alpha)} - c\epsilon_{(\beta)} - d\epsilon_{(\gamma)}}{5 - c - d} \quad (2.73)
 \end{aligned}$$

Using this procedure we can determine all the $r_{(\alpha_i)}$ in the counterterm expression (2.71).

At this point the analysis of the theory falls into two possible classes.

- The first class is to analyze the theory right at the usual boundary where $r_c \rightarrow \infty$. This is the standard picture where there are infinite degrees of freedom at the boundary, and the theory has a smooth RG flow from UV to IR till it confines (at least from the weakly coupled gravity dual). The action obtained by setting $r_c = \infty$ describes the ‘parent cascading theory’.
- The second class is to analyze the theory by specifying the degrees of freedom at generic energy scale given by $r_c = \Lambda_c$ and then defining the theories at the boundary by adding appropriate irrelevant and marginal operators for scales greater than Λ_c .

By adding different operator for scales $\Lambda > \Lambda_c$, one gets different theories. All these different branches meet the parent cascading theory at some scale $r_c = \Lambda_c$, as depicted in Fig **2.13**. The gravity duals of these theories are the usual *deformed* conifold geometries cutoff at various r_c with appropriate UV caps added (of course for $r < r_c$ the geometries change accordingly). More details on UV caps to geometries will be discussed in the next subsection.

The first class of theories is more relevant for the pure AdS/CFT case whereas the latter is more relevant for the non-AdS case⁸. For the pure AdS/CFT case without flavors $\epsilon_{(\alpha_i)} = 0$ (so that the subscript α_i 's can be ignored from all variables), we can subtract the counter-terms (2.71) from the action (2.68) to get the following renormalized action:

$$\begin{aligned}\mathcal{S}_{\text{ren}}^{(1)} &= \mathcal{S}^{(1)} - \mathcal{S}_{\text{counter}}^{(1)} \\ &= \int \frac{d^4 q}{(2\pi)^4} \left[H^{mn} \left(s_{mm}^{(4)} s_{nn}^{(0)} + s_{mm}^{(3)} s_{nn}^{(1)} + s_{mm}^{(2)} s_{nn}^{(2)} + s_{mm}^{(1)} s_{nn}^{(3)} \right. \right. \\ &\quad \left. \left. + s_{mm}^{(0)} s_{nn}^{(4)} \right) - K^{mn} \left(4s_{mm}^{(0)} s_{nn}^{(4)} + 3s_{mm}^{(1)} s_{nn}^{(3)} + 4s_{mm}^{(2)} s_{nn}^{(2)} + s_{mm}^{(3)} s_{nn}^{(1)} \right. \right. \\ &\quad \left. \left. 4s_{nn}^{(0)} s_{mm}^{(4)} + 3s_{nn}^{(1)} s_{mm}^{(3)} + s_{nn}^{(3)} s_{mm}^{(1)} \right) + I^m s_{mm}^{(4)} \right] \end{aligned} \quad (2.74)$$

where we have made all the $\mathcal{O}(1/r_c)$ terms vanishing, and in the limit r_h small the small shifts to $s_{nn}^{(j)}$ given by $s_{nn}^{(3)} + \mathcal{O}(r_h^4)$ can also be ignored. Observe that we can reinterpret the renormalized action (2.74) as the following new action:

$$\begin{aligned}\mathcal{S}_{\text{ren}}^{(1)} &= \int \frac{d^4 q}{(2\pi)^4} \left[Z^{mn} \Phi_m(q) \Phi_n(-q) + U^{mn} (\Phi_m(q) \Phi'_n(-q) + \Phi'_m(q) \Phi_n(-q)) \right. \\ &\quad \left. + Y^m \Phi_m(q) + Y^n \Phi_n(-q) + V^m \Phi'_m(q) + V^n \Phi'_n(-q) + X \right] \end{aligned} \quad (2.75)$$

where $\Phi_m(q) = \Phi_m(q, r = \infty)$, $\Phi'_m(q) = d\Phi_m(q, r)/dr|_{r=\infty}$ and X, Y, Z, U, V could be functions of r and \vec{q} but evaluated for fixed $r = \infty$. We can determine their functional form by comparing (2.75) with (2.74). For us however the most relevant

⁸In both cases of course we need to add appropriate number of seven branes to get the finite F-theory picture. The holographic renormalization procedure remains unchanged and the far IR physics remains unaltered. The UV caps affect mostly geometries close to r_c , as expected.

part is the energy momentum tensors which we could determine from 2.75 by finding the coefficients Y^m and Y^n . One can easily show that, up to a possible additive constant, Y^m, Y^n are given by:

$$\begin{aligned} Y^m &= H^{mn} s_{nn}^{(4)} - 4K^{mn} s_{nn}^{(4)}, & Y^n &= H^{mn} s_{mm}^{(4)} - 4K^{mn} s_{mm}^{(4)} \\ V^m &= K^{mn} s_{nn}^{(5)}, & V^n &= K^{mn} s_{mm}^{(5)} \end{aligned} \quad (2.76)$$

Now let us come to second class of theories wherein we take any arbitrary $r = r_c$, with appropriate UV degrees of freedom such that they have good boundary descriptions satisfying all the necessary constraints. For these cases, once we subtract the counter-terms (2.71), the renormalized action (specified by r) takes the following form:

$$\begin{aligned} \mathcal{S}_{\text{ren}}^{(1)} &= \mathcal{S}^{(1)} - \mathcal{S}_{\text{counter}}^{(1)} \\ &= \int \frac{d^4 q}{2(2\pi)^4} \sum_{\alpha, \beta, \gamma} \left\{ H_{|\alpha|}^{mn} \left(s_{mm}^{(4)[\beta]} s_{nn}^{(0)[\gamma]} r^{4\epsilon_{(\beta)} - 4\epsilon_{(\alpha)}} + s_{mm}^{(3)[\beta]} s_{nn}^{(1)[\gamma]} r^{3\epsilon_{(\beta)} + \epsilon_{(\gamma)} - 4\epsilon_{(\alpha)}} \right) \right. \\ &\quad + s_{mm}^{(2)[\beta]} s_{nn}^{(2)[\gamma]} r^{2\epsilon_{(\beta)} + 2\epsilon_{(\gamma)} - 4\epsilon_{(\alpha)}} + s_{mm}^{(1)[\beta]} s_{nn}^{(3)[\gamma]} r^{\epsilon_{(\beta)} + 3\epsilon_{(\gamma)} - 4\epsilon_{(\alpha)}} + s_{mm}^{(0)[\beta]} s_{nn}^{(4)[\gamma]} r^{4\epsilon_{(\gamma)} - 4\epsilon_{(\alpha)}} \Big) \\ &\quad - 4K_{|\alpha|}^{mn} \left(s_{mm}^{(0)[\beta]} s_{nn}^{(4)[\gamma]} r^{5\epsilon_{(\gamma)} - 5\epsilon_{(\alpha)}} + s_{mm}^{(1)[\beta]} s_{nn}^{(3)[\gamma]} [r^{\epsilon_{(\beta)} + 4\epsilon_{(\gamma)} - 5\epsilon_{(\alpha)}} + r^{2\epsilon_{(\beta)} + 3\epsilon_{(\gamma)} - 5\epsilon_{(\alpha)}}] \right. \\ &\quad + 4s_{mm}^{(2)[\beta]} s_{nn}^{(2)[\gamma]} [r^{2\epsilon_{(\beta)} + 3\epsilon_{(\gamma)} - 5\epsilon_{(\alpha)}} + r^{3\epsilon_{(\beta)} + 2\epsilon_{(\gamma)} - 5\epsilon_{(\alpha)}}] + 4s_{nn}^{(0)[\beta]} s_{mm}^{(4)[\gamma]} r^{5\epsilon_{(\beta)} - 5\epsilon_{(\alpha)}} \\ &\quad \left. + s_{mm}^{(3)[\beta]} s_{nn}^{(1)[\gamma]} [r^{4\epsilon_{(\beta)} + \epsilon_{(\gamma)} - 5\epsilon_{(\alpha)}} + r^{3\epsilon_{(\beta)} + 2\epsilon_{(\gamma)} - 5\epsilon_{(\alpha)}}] \right) + I_{|\alpha|}^m s_{mm}^{(4)[\beta]} r^{5\epsilon_{(\beta)} - 5\epsilon_{(\alpha)}} \Big\} \quad (2.77) \end{aligned}$$

evaluated at the cut-off r_c and the horizon radii r_h as usual. Notice now the appearance of $r^{m\epsilon_{(\alpha)} + n\epsilon_{(\beta)} + p\epsilon_{(\gamma)}}$ factors. One can easily show that:

$$\frac{1}{2} [r^{m\epsilon_{(\alpha)} + n\epsilon_{(\beta)} + p\epsilon_{(\gamma)}} + r^{-m\epsilon_{(\alpha)} - n\epsilon_{(\beta)} - p\epsilon_{(\gamma)}}] = 1 + \mathcal{O}[\epsilon_{(\alpha, \beta, \gamma)}]^2 \quad (2.78)$$

Since the warp factor h is defined only for small values of $g_s N_f, g_s M^2/N, g_s^2 N_f M^2/N$ we don't know the background (and hence the warp factor) for finite values of these quantities. Therefore for our case we can put $\mathcal{O}[\epsilon_{(\alpha, \beta, \gamma)}]^2$ to zero so that the value in (2.78) is identically 1. For finite values of these quantities both the warp factor and the background would change drastically and so new analysis need to be performed to holographically renormalize the theory. Our conjecture would be that once we know the background for finite values of $g_s N_f, g_s M^2/N$, the terms like (2.77) would come out automatically renormalized by choice of our counterterms.

Once this is settled, the renormalized action at the cut-off radius r_c would only go as powers of $r_{c(\alpha)}^{-1}$. Thus we can express the total action as:

$$\begin{aligned} \mathcal{S}_{\text{ren}}^{(1)} = \int \frac{d^4 q}{(2\pi)^4} \sum_{\alpha, \beta} \left\{ \left(\sum_{j=0}^{\infty} \frac{\tilde{a}_{mn(j)}^{(\alpha)}}{r_{c(\alpha)}^j} \right) \tilde{G}^{mn} \Phi_m(q) \Phi_n(-q) + \left(\sum_{j=0}^{\infty} \frac{\tilde{e}_{mn(j)}^{(\alpha)}}{r_{c(\alpha)}^j} \right) \tilde{M}^{mn} (\Phi_m(q) \Phi'_n(-q) \right. \\ \left. + \Phi'_m(q) \Phi_n(-q)) + H_{|\alpha|}^{mn} [s_{nn}^{(4)[\beta]} \Phi_m(q) + s_{mm}^{(4)[\beta]} \Phi_n] + K_{|\alpha|}^{mn} [-4s_{nn}^{(4)[\beta]} \Phi_m(q) - 4s_{mm}^{(4)[\beta]} \Phi_n(q) \right. \\ \left. + s_{nn}^{(5)[\beta]} \Phi'_m(q) + s_{mm}^{(5)[\beta]} \Phi'_n(q)] + \left(\sum_{j=0}^{\infty} \frac{\tilde{b}_{m(j)}^{(\alpha)}}{r_{c(\alpha)}^j} \right) \tilde{J}^m \Phi_m(q) + X[r_{c(\alpha)}] \right\} \left[1 - \frac{r_h^4}{r_c^4} \right]^{-\frac{1}{2}} \quad (2.79) \end{aligned}$$

where Φ_n are as before and $r_{c(\alpha)}^j = r_{c(\alpha)}^j|_{r=r_c}$. Here we have ignored the terms evaluated at the horizon as they are not proportional to the sources Φ_n and hence will not contribute to the expectation value. The explicit expressions for the other coefficients listed above, namely, \tilde{G}^{mn} , \tilde{M}^{mn} , $\tilde{a}_{mn(j)}^{(\alpha)}$, \tilde{J}^m , $\tilde{e}_{mn(j)}^{(\alpha)}$ and $\tilde{b}_{m(j)}^{(\alpha)}$ can be worked out easily from our earlier analysis (see Appendix C of [57] for one specific example). Note that $X[r_{c(\alpha)}]$ is a function independent of $\Phi_m^{[0]}$ and appears for generic renormalized action.

Now the generic form for the energy momentum tensor is evident from looking at the linear terms in the above action (2.79). This is then given by:

$$\begin{aligned} T_0^{mm} \equiv \int \frac{d^4 q}{(2\pi)^4} \left[(H_{|\alpha|}^{mn} + H_{|\alpha|}^{nm}) s_{nn}^{(4)[\beta]} - 4(K_{|\alpha|}^{mn} + K_{|\alpha|}^{nm}) s_{nn}^{(4)[\beta]} + (K_{|\alpha|}^{mn} + K_{|\alpha|}^{nm}) s_{nn}^{(5)[\beta]} \right. \\ \left. + \left(\sum_{j=0}^{\infty} \frac{\tilde{b}_{n(j)}^{(\alpha)}}{r_{c(\alpha)}^j} \right) \tilde{J}^n \delta_{nm} \right] \left(1 - \frac{r_h^4}{r_c^4} \right)^{-\frac{1}{2}} \quad (2.80) \end{aligned}$$

at $r = r_c$ (we ignore the result at the horizon) and sum over (α, β) is implied. This result should be compared to the ones derived in [63] [64][65] [66][67] [68] [69] which don't have any r_c dependence. The AdS/CFT result is of course the first line of the above result. Notice that the second line also has a r_c independent additive constant which are irrelevant for our purpose because the energy-momentum can always be shifted by a constant to absorb this factor. Now for non-AdS geometries, r_c determines the number of degrees of freedom at scale Λ_c with the relation (3.33) and inverting it with $r^4 h \sim \log(r)$ we get $r_c \sim e^{N_{\text{eff}}}$. Thus once we specify the effective degrees of freedom at scale Λ_c , then our result shows that the energy-momentum tensor not only inherits the universal behavior of the parent cascading theory but

there are additional corrections of $\mathcal{O}(e^{-jN_{\text{eff}}})$. But energy scale should eventually be set to infinity, so how is keeping $\mathcal{O}(1/r_c)$ terms in (2.80) justified?

To answer this, first consider a daughter gauge theory for which number of degrees of freedom do not change for scales $\Lambda > \Lambda_c$ and then $N_{\text{eff}}(\Lambda_c) = N_{\text{UV}}(\infty)$. Taking the energy scale to infinity means replacing r_c in (2.80) with $e^{N_{\text{UV}}(\infty)}$ as only gauge theory observables such as number of degrees of freedom should appear in the final result for stress tensor. But as $N_{\text{eff}}(\Lambda_c) = N_{\text{UV}}(\infty)$, for this particular theory, we can keep the corrections coming from $\mathcal{O}(1/r_c)$ as in (2.80).

This argument of replacing $r_c \rightarrow \mathcal{O}(e^{N_{\text{UV}}})$ is somewhat naive and needs elaboration. If we cut off the geometry at $r = r_c$, and *do not* add any geometry from $r = r_c$ to $r = \infty$, we are essentially ignoring gravitons that are present in the region $r \geq r_c$. This will in turn mean we are ignoring UV modes in the dual field theory that have energies $\Lambda \geq \Lambda_c$. Thus (2.80) is incomplete. What we mean here is once we add gravitons from region $r \geq r_c$, their contribution to the final stress energy tensor will can be accounted for by replacing r_c in (2.80) with $\mathcal{O}(e^{N_{\text{UV}}})$.

Note that for an AdS geometry, the dual gauge theory is conformal and do not change degrees of freedom with scale. Hence treating $N_{\text{eff}}(\Lambda_c) = N_{\text{UV}}(\infty)$ amounts to adding an AdS UV cap to our dual geometry from $r = r_c$ to $r = \infty$. The final result for stress tensor of a dual gauge theory with an AdS UV geometry will be of the form (2.80) with $r_c \rightarrow \mathcal{O}(e^{N_{\text{UV}}})$ which may not be infinity. We will discuss this in detail in the following section.

In general, various gauge theories which are quite distinct at far UV scale can look identical in the IR. One can start with a given number of degrees of freedom at a scale Λ_c and demand that it is the infrared limit of several different UV theories. Finally specifying the UV degrees of freedom distinguishes the theories. From the gravity side we can replace r_c in (2.80) with different values of $e^{N_{\text{UV}}}$ and obtain distinct UV gauge theories which are identical at the infrared scale Λ_c . In fact a more careful analysis shows that one can cut a bulk geometry at some r_c and attach various UV geometries from r_c to ∞ to obtain distinct gauge theories at the boundary. We discuss this in some detail in the following section.

2.3.2 UV Completion from Dual Gravity

The first important issue here is that we can study infinite number of UV completed theories in full F-theory setup. All of these theories have good boundary descriptions and have same degrees of freedom as the parent cascading theory at certain specified scales. The simplest UV complete theory is of course the parent cascading theory of OKS model. Using (3.33) with warp factor given by (2.31) one easily gets

$$N_{\text{eff}}(\Lambda) \sim L^4(1 + \mathcal{O}(\log(\Lambda))) \quad (2.81)$$

which grows with scale Λ and diverges for $\Lambda \rightarrow \infty$. The question now is how to construct other possible theories by defining the degrees of freedom at scale $\Lambda \rightarrow \infty$ i.e. at the boundary of the geometry $r \rightarrow \infty$. The action for the boundary theory should be identified as:

$$[\mathcal{S}_{\text{ren}}^{(1)}]_{r_h}^\infty = [\mathcal{S}_{\text{ren}}^{(1)}]_{r_h}^{r_c} + [\mathcal{S}_{\text{ren}}^{(1)}]_{r_c}^\infty \quad (2.82)$$

where the boundary is at $r \rightarrow \infty$. For the boundary cascading theory the above expression simply means that

$$\begin{aligned} [\mathcal{S}_{\text{ren}}^{(1)}]_{r_c}^\infty &= - \int \frac{d^4 q}{(2\pi)^4} \left(\sum_{j=0}^\infty \frac{\tilde{b}_{n(j)}^{(\alpha)}}{r_{c(\alpha)}^j} \right) \tilde{J}^n \Phi_n - \int \frac{d^4 q}{(2\pi)^4} \left(\sum_{j=0}^\infty \frac{B_{n(j)}^{(\alpha)} r_h^{4j}}{r_{c(\beta)}^j} \right) \Phi_n \\ &\quad + \frac{\{\tilde{b}_{n(j)}^{(\alpha)}, \mathcal{O}(r_h^{4j})\}}{\infty} \text{ factors} \end{aligned} \quad (2.83)$$

where the sign is crucial and sum over α is again implied (note (a) the cut-off dependence, and (b) $r_{c(\beta)}$ is some function of $r_{c(\alpha)}$ that one can determine easily). We now see that the contributions from the UV cap give the following values for B_j for parent cascading theory:

$$\begin{aligned} B_{n(0)}^{(\alpha)} &= B_{n(1)}^{(\alpha)} = B_{n(2)}^{(\alpha)} = B_{n(3)}^{(\alpha)} = 0 \\ B_{n(4)}^{(\alpha)} &= \frac{1}{2} \left[\tilde{b}_{n(0)}^{(\alpha)} \tilde{J}^n \theta(r_0 - r_c) + \mathcal{O}(H_{|\alpha|}^{nm}, K_{|\alpha|}^{nm}, s_{nn}^{[\alpha]}) \right], \dots \end{aligned} \quad (2.84)$$

The choice above is made precisely to *exactly* cancel the $\mathcal{O}(1/r_c)$ contributions coming in from the action measured from $r_h \leq r \leq r_c$. Finally the boundary energy-

momentum tensor reads

$$\int \frac{d^4 q}{(2\pi)^4} \sum_{\alpha, \beta} \left[(H_{|\alpha|}^{mn} + H_{|\alpha|}^{nm}) s_{nn}^{(4)[\beta]} - 4(K_{|\alpha|}^{mn} + K_{|\alpha|}^{nm}) s_{nn}^{(4)[\beta]} + (K_{|\alpha|}^{mn} + K_{|\alpha|}^{nm}) s_{nn}^{(5)[\beta]} \right] \quad (2.85)$$

which is the result derived in [63]-[67] and [69].

The above way of reinterpreting the boundary contribution should tell us precisely how we could modify the boundary degrees of freedom to construct distinct UV completed theories. There are two possible ways we can achieve this:

- From the geometrical perspective we can cutoff the deformed conifold background at $r = r_c$ and attach an appropriate UV “cap” from $r = r_c$ to $r \rightarrow \infty$ by carefully modifying the geometry at the neighborhood of the junction point. As an example, this UV cap could as well be another AdS background from r_c to $r \rightarrow \infty$. There are of course numerous other choices available from the F-theory limit. Each of these caps would give rise to distinct UV completed gauge theories.
- From the action perspective we could specify the value of the action measured from r_c to $r \rightarrow \infty$, i.e $[\mathcal{S}_{\text{ren}}^{(1)}]_{r_c}^\infty$. The simplest case where this is zero gives rise to a boundary theory for which $N_{\text{eff}}(\Lambda_c) = N_{\text{UV}}(\infty)$ as discussed before. To study more generic cases, we need to see how much constraints we can put on our integral. One immediate constraint is the holographic renormalizability of our theory. This tells us that the value of the integral can only go as powers of $1/r_{c(\alpha)}$ otherwise we will not have finite actions. This in turn implies

$$\begin{aligned} [\mathcal{S}_{\text{ren}}^{(1)}]_{r_c}^\infty &= \int \frac{d^4 q}{(2\pi)^4} \sum_{\alpha} \left\{ \left(\sum_{j=0}^{\infty} \frac{\tilde{A}_{mn(j)}^{(\alpha)}}{r_{c(\alpha)}^j} \right) \tilde{G}^{mn} \Phi_m \Phi_n + \left(\sum_{j=0}^{\infty} \frac{\tilde{E}_{mn(j)}^{(\alpha)}}{r_{c(\alpha)}^j} \right) \tilde{M}^{mn} \right. \\ &\quad \left. \times (\Phi_m \Phi'_n + \Phi'_m \Phi_n) + \left(\sum_{j=0}^{\infty} \frac{\tilde{B}_{m(j)}^{(\alpha)}}{r_{c(\alpha)}^j} \right) \tilde{J}^m \Phi_m + X[r_{(\alpha)}] \right\} \theta(r_0 - r_c) + \text{finite terms} \end{aligned} \quad (2.86)$$

where by specifying the coefficients $\tilde{A}_{mn(j)}^{(\alpha)}$, $\tilde{E}_{mn(j)}^{(\alpha)}$ and $\tilde{B}_{m(j)}^{(\alpha)}$ we can specify the precise UV degrees of freedom! The finite terms are r_c independent and therefore would only provide finite shifts to our observables. They could therefore be scaled to zero. Notice also that the contributions from (2.86) only renormalizes the coefficients $\tilde{a}_{mn(j)}^{(\alpha)}$, $\tilde{e}_{mn(j)}^{(\alpha)}$ and $\tilde{b}_{m(j)}^{(\alpha)}$ in (2.79), and therefore the final expressions for all the physical variables

for various UV completed theories could be written directly from (2.79) simply by replacing the $1/r_c$ dependent coefficients by their renormalized values. This is thus our precise description of how to specify the UV degrees of freedom for various gauge theories in our setup (see Fig 2.15 below).

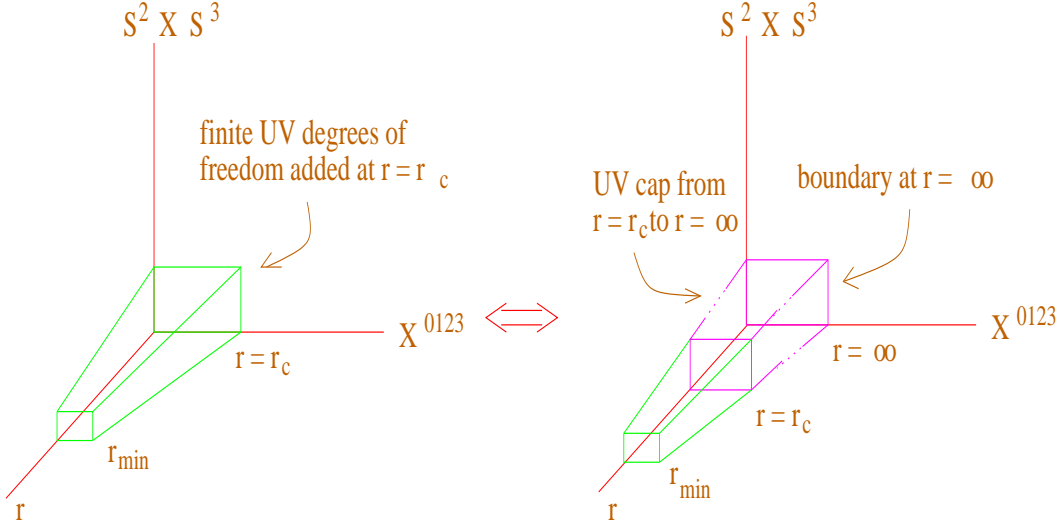


Figure 2.15: The equivalence between two different ways of viewing the boundary theory at zero temperature.

For the left figure in 2.15 we add finite UV degrees of freedom at $r = r_c$ of the deformed conifold geometry. Such a process is equivalent to the figure on the right where we cut-off the deformed conifold geometry at $r = r_c$ and add a UV cap from $r = r_c$ to $r = \infty$. The boundary theory on the right has \mathcal{N}_{uv} degrees of freedom at $r = \infty$ and all physical quantities computed in either of these two pictures would only depend on \mathcal{N}_{uv} but not on $r = r_c$. At non-zero temperature the UV descriptions remain unchanged.

Once the UV descriptions are properly laid out, we can determine the form for $\tilde{A}^{(\alpha)}$, $\tilde{B}^{(\alpha)}$ and $\tilde{E}^{(\alpha)}$ by writing Callan-Symanzik type equations for them. They tell us how $\tilde{A}^{(\alpha)}$, $\tilde{B}^{(\alpha)}$ and $\tilde{E}^{(\alpha)}$ would behave with the scale r_c or equivalently μ_c . For \tilde{A}

the equation is⁹:

$$\mu_c \frac{\partial \tilde{A}_{mn(j)}^{(\alpha)}}{\partial \mu_c} = j [1 - \epsilon_{(\alpha)}] [\tilde{A}_{mn(j)}^{(\alpha)} + \tilde{a}_{mn(j)}^{(\alpha)}] \quad (2.87)$$

with similar equations for $\tilde{B}^{(\alpha)}$ and $\tilde{E}^{(\alpha)}$. These equations tell us that physical quantities are independent of scales. The parent cascading theory is defined as the scale-invariant limits of (2.87), i.e:

$$\tilde{A}_{mn(j)}^{(\alpha)} = -\tilde{a}_{mn(j)}^{(\alpha)}, \quad \tilde{E}_{mn(j)}^{(\alpha)} = -\tilde{e}_{mn(j)}^{(\alpha)}, \quad \tilde{B}_{m(j)}^{(\alpha)} = -\tilde{b}_{m(j)}^{(\alpha)} \quad (2.88)$$

The above relation gives us a hint how to express $\tilde{A}^{(\alpha)}$, $\tilde{B}^{(\alpha)}$ and $\tilde{E}^{(\alpha)}$ in terms of \mathcal{N}_{eff} , the effective degrees of freedom at $r = r_c$ and \mathcal{N}_{uv} , the effective degrees of freedom at $r = \infty$ i.e the boundary:

$$\begin{aligned} \tilde{A}_{mn(j)}^{(\alpha)} &= -\tilde{a}_{mn(j)}^{(\alpha)} + \hat{a}_{mn(j)}^{(\alpha)} e^{-j[\mathcal{N}_{\text{uv}} - (1 - \epsilon_{(\alpha)})\mathcal{N}_{\text{eff}}]} \\ \tilde{E}_{mn(j)}^{(\alpha)} &= -\tilde{e}_{mn(j)}^{(\alpha)} + \hat{e}_{mn(j)}^{(\alpha)} e^{-j[\mathcal{N}_{\text{uv}} - (1 - \epsilon_{(\alpha)})\mathcal{N}_{\text{eff}}]} \\ \tilde{B}_{m(j)}^{(\alpha)} &= -\tilde{b}_{m(j)}^{(\alpha)} + \hat{b}_{m(j)}^{(\alpha)} e^{-j[\mathcal{N}_{\text{uv}} - (1 - \epsilon_{(\alpha)})\mathcal{N}_{\text{eff}}]} \end{aligned} \quad (2.89)$$

where the actual boundary degrees of freedom are specified by knowing $\hat{a}_{mn(j)}$, $\hat{e}_{mn(j)}$ and $\hat{b}_{m(j)}$ as well as \mathcal{N}_{uv} . Since j goes from 0 to ∞ , there are infinite possible UV complete boundary theories possible¹⁰. For very large \mathcal{N}_{uv} (i.e $\mathcal{N}_{\text{uv}} \rightarrow \epsilon^{-n}, n \gg 1$) the boundary theories are similar to the original cascading theory. The various choices of $(\hat{a}_{mn(j)}^{(\alpha)}(\vec{q}), \hat{e}_{mn(j)}^{(\alpha)}(\vec{q}), \hat{b}_{m(j)}^{(\alpha)}(\vec{q}))$ tell us how the degrees of freedom change from \mathcal{N}_{uv} to \mathcal{N}_{eff} under RG flow. The \vec{q} dependence of all the quantities will tell us how the UV degrees of freedom affect IR physics. This is to be expected: addition of irrelevant operators do change IR physics, but not the far IR¹¹.

⁹The following equation is derived from the scale-invariance of $[\mathcal{S}_{\text{ren}}^{(1)}]_{r_h}^\infty$.

¹⁰The connection of j with UV completions come from the coefficients $\hat{a}_{mn(j)}^{(\alpha)}$, $\hat{e}_{mn(j)}^{(\alpha)}$ and $\hat{b}_{m(j)}^{(\alpha)}$ etc. that depend on j . For different choices of these coefficients we can have different UV completions. In this sense j and UV completions are related.

¹¹The \vec{q} dependences of the UV caps are also one-to-one correspondence to the changes in the local geometries near the cut-off radius r_c , as we discussed before. All in all this fits nicely with what one would have expected from UV degrees of freedom. Of course it still remains to verify the

Therefore with this understanding of the boundary theories we can express the energy-momentum tensor at the boundary with \mathcal{N}_{uv} degrees of freedom at the boundary purely in terms of gauge theory variables, as:

$$T_0^{mm} \equiv \int \frac{d^4 q}{(2\pi)^4} \left[(H_{|\alpha|}^{mn} + H_{|\alpha|}^{nm}) s_{nn}^{(4)[\beta]} - 4(K_{|\alpha|}^{mn} + K_{|\alpha|}^{nm}) s_{nn}^{(4)[\beta]} + (K_{|\alpha|}^{mn} + K_{|\alpha|}^{nm}) s_{nn}^{(5)[\beta]} \right. \\ \left. + \sum_{j=0}^{\infty} \hat{b}_{n(j)}^{(\alpha)}(\vec{q}) \tilde{J}^n e^{-j\mathcal{N}_{uv}} \delta_{nm} \right] \quad (2.90)$$

where sum over α is again implied, and the first line is the universal property of the parent cascading theory inherited by our gauge theory. The second line specifies the precise degrees of freedom that we add at $r = r_c$ to describe the UV behavior of our theory at the boundary $r \rightarrow \infty$. Using this procedure, the final results of any physical quantities should be expressed only in terms of \mathcal{N}_{uv} i.e the UV degrees of freedom¹².

2.4 Towards Large N QCD from Dual Gravity

From the above discussions we see how one obtains gauge theories which are quite distinct in the UV but have common IR dynamics. With a clear understanding of how to UV complete a gauge theory with particular properties in the IR, we can construct a brane configuration that mimics large N QCD.

First observe from (3.33) and (2.81) that in OKS model the number of degrees of freedom keep on growing in the UV. Furthermore with $B_2 \sim \log(r)$ and axio-dilaton behaving as in (2.41) we see that the gauge couplings run as $g_i \sim \log(\Lambda)$

story from an actual supergravity calculation. We need to analyze the metric near the junction by studying the continuity and differentiability of the metric and see how far below $r = r_c$ we expect deformations from the UV caps. Various types of deformations will signal various sets of irrelevant operators. Needless to say, the far IR physics remain completely unaltered. In the following section, we will perform an exact calculation using appropriate sources in SUGRA action accounting the deformation in the region near the junction.

¹²Restoring back the $\mathcal{O}(r_h)$ contributions would mean that there should be an additional contribution to (2.90) of the form $\sum_{j=0}^{\infty} G(\hat{b}_n^{(\alpha)}, H_{|\alpha|}^{mn}, K_{|\alpha|}^{mn}, s_{nn}^{[\alpha]}) \mathcal{T}^{4j} e^{-j\mathcal{N}_{uv}}$ where G is a function whose functional form could be inferred from the UV integral (2.86).

using (2.21). In fact in the far UV, at least one of the two gauge couplings blow up and we have Landau poles. Thus the UV of OKS model has runaway behavior while QCD is asymptotically a free theory [70][71]. However the gauge couplings do run logarithmically and in the far IR using a cascade of Seiberg dualities one obtains $SU(\bar{M})$ gauge group with fundamental matter. Hence the IR of OKS model has common features with large N QCD while the UV is quite different. But as discussed in the previous section, we can modify the UV of OKS model and build dual gravity of a gauge theory that resembles large N QCD more closely.

The ideal scenario would be to build a gauge theory which becomes asymptotically free and for scales where coupling is large, it has a description in terms of weakly coupled classical gravity. Currently there are no brane setups that realizes such running of coupling and has dual gravity in the appropriate regime. What we have been able to construct in [62] is a gauge theory that becomes asymptotically *conformal* in the far UV and has logarithmic running of couplings in the IR. The regime where the gauge theory has a classical supergravity description is when gauge coupling is large and thus from UV to IR we have strong but running gauge coupling. This gauge theory has no Landau poles and the number of degrees of freedom do not diverge as $\Lambda \rightarrow \infty$. In this section we will briefly describe the brane setup that gives rise to such a gauge theory and construct its dual gravity. Details of our model can be found in [62].

We start with the brane setup of Fig 2.11 i.e. the OKS model. The logarithmic running of the couplings come from the global logarithmic running of NS-NS two form $B_2 \sim \log(r)$ and the *local* logarithmic running of axio-dilaton near a seven brane as in (2.41). On the other hand the three form fluxes run as $1/r$, i.e. similar to the scalar potential for a single charge located at $r = 0$. By introducing anti charge at some location $r \sim r_0$, we can get the total three form flux to behave like $1/r^2$ for large $r \gg r_0$ i.e. similar to scalar potential of a dipole. This would result in total NS-NS two form $B_2 \sim 1/r$ for $r \gg r_0$. We can also arrange 7 branes in such a way that F theory gives $\tau \sim \mathcal{O}(1/r^n)$ for large $r \gg r_0$ with $n > 0$. Details of the 7 brane embedding and the global behavior of τ will be discussed in section 2.4.3. The anti-

charges we introduce are anti five branes in the neighborhood of $r \sim r_0$. Of course anti branes want to slide down to the tip of the conifold and we need to introduce sufficient flux to keep them at $r \sim r_0$. The resulting brane setup is depicted in Fig 2.16.

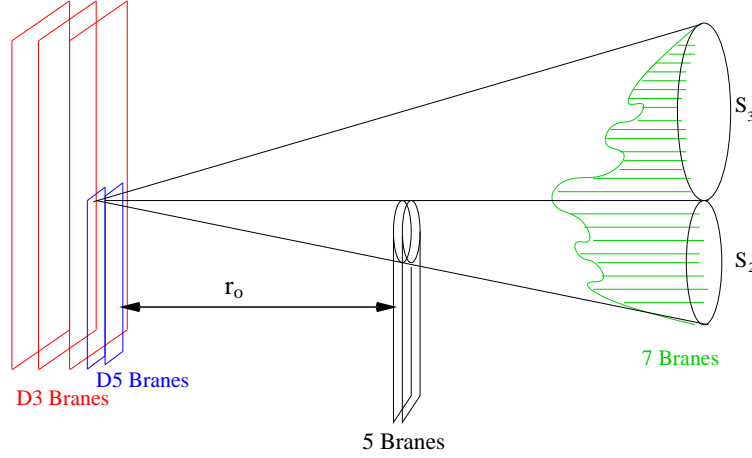


Figure 2.16: Brane configuration for asymptotic conformal gauge theory.

The dual gravity for the brane setup in Fig 2.16 considering non-zero temperature is sketched in Fig 2.17. The geometry splits into three regions of interest: Regions 1, 2 and 3. The figure shows the various regions of interest. As should be clear, most of the seven branes lie in Region 3, except for a small number of coincident seven branes that dip till r_{\min} i.e Region 1. The interpolating region is Region 2. Region 1 is basically the one discussed in great details in [57] and the previous sections. In this region there is one (or a coincident set of) seven brane(s) along with $B_2 \sim \log(r)$. The logarithmic dependences of the warp factor, fluxes and subsequently of the gauge couplings originate from B_2 and these coincident (or single) seven branes.

The UV cap in the full F-theory framework is depicted as Region 3 in the above figure. In this region the seven branes are distributed so that axio-dilaton has the behavior i.e. $\tau \sim \mathcal{O}(1/r^n), n \geq 0$ while NS-NS two form B_2 go as $\mathcal{O}(1/r^m), m \geq 0$. Using these precise forms in (2.21) one gets that the gauge couplings go as $g_i \sim \mathcal{O}(1/r^m) = \mathcal{O}(1/\Lambda^m)$, i.e. we have asymptotic AdS space giving almost conformal

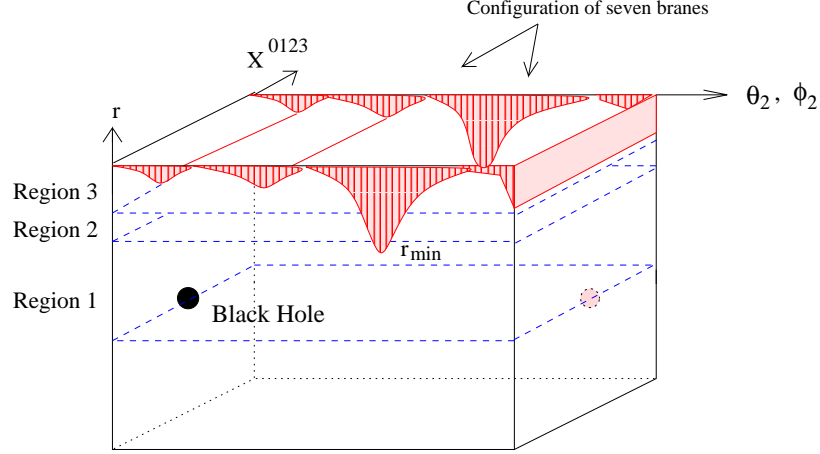


Figure 2.17: Dual geometry for brane setup in Fig 2.16

field theory in the far UV.

One big advantage about our UV cap is related to the issues raised in [72]. Since the H_{NS} and the axio-dilaton fields have well defined behaviors at large r , there would be *no* UV divergences of the Wilson loops in our picture! Therefore our configuration can not only boast of holographic renormalizability, but also of the absence of Landau poles and the associated UV divergences of the Wilson loops. We will have more to say about Wilson loops in section 3.3.

The metric in all three regions can be written in the following form [62]

$$ds^2 = \frac{1}{\sqrt{h}} \left[-g_1 dt^2 + dx^2 + dy^2 + dz^2 \right] + \sqrt{h} \left[g_2^{-1} g_{rr} dr^2 + g_{mn} dx^m dx^n \right] \quad (2.91)$$

with g_i being the Black-Hole factors and h being the warp factor depends on all the internal coordinates $(r, \theta_i, \phi_i, \psi)$. To zeroth order in $g_s N_f$ and $g_s M$ we have our usual relations:

$$h^{[0]} = \frac{L^4}{r^4}, \quad g^{[0]} = 1 - \frac{r_h^4}{r^4}, \quad g_{rr}^{[0]} = 1, \quad g_{mn}^{[0]} dx^m dx^n = ds_{T^{11}}^2 \quad (2.92)$$

But in higher order in $g_s N_f, g_s M$, both the warp factor and the internal metric get modified because of the back-reactions from the seven-branes, three form fluxes and the localized sources we embed. We can write this as:

$$h = h^{[0]} + h^{[1]}, \quad g_{rr} = g_{rr}^{[0]} + g_{rr}^{[1]}, \quad g_{mn} = g_{mn}^{[0]} + g_{mn}^{[1]} \quad (2.93)$$

where the superscripts denote the order of $g_s N_f$ and $g_s M$. Now observe that to linear order in $g_s N_f$ and $g_s M$, $g_{rr}^{[1]} = g_{mn}^{[1]} = 0$ and we recover OKS metric (2.23) with warp factor given by (2.31) in Region 1. Thus only when we include higher order terms, the internal metric deforms from the metric of $T^{1,1}$. Whereas for a given order in $g_s N_f, g_s M$ the warp factor h behaves logarithmically for small r and as $1/r^m$ for large r .

It is clear that one cannot jump from Region 1 to Region 3 abruptly. There should be an interpolating geometry where fluxes and the metric should have the necessary property of connecting the two solutions. This is Region 2 in our figure above.

In the following, we briefly discuss the backgrounds for all the three regions. For a comprehensive analysis, please consult our paper [62].

2.4.1 Region 1: Fluxes, Metric and the Coupling Constants Flow

As already mentioned, the metric of the entire geometry has the form given in (2.91) with warp factor h given by (2.31) in region 1. The internal space retains its resolved-deformed conifold form up to $\mathcal{O}(g_s N_f)$. Beyond this order the internal space loses its simple form and becomes a complicated non-Kähler manifold. The background has *all* the type IIB fluxes switched on, with the three-forms given by (2.39), (2.40) while five-form and the axio-dilaton are given by (2.41).

2.4.2 Region 2: Interpolating Region and the Detailed Background

To attach a UV cap that allows conformal invariance i.e. vanishing beta function, we need at least a configuration of vanishing NS three-form and axio-dilaton that becomes constant as $r \rightarrow \infty$. This cannot be *abruptly* attached to Region 1: we need an interpolating region. This region, which we will call Region 2, should have the behavior that at the outermost boundary the three-forms vanish, while solving the equations of motion. The innermost boundary of Region 2 – that also forms the

outermost boundary of Region 1 – will be determined by the scale associated with the mass of the lightest quark, m_0 , in our system. In Fig **2.17**, this is given by region in the local neighborhood of $r_{\min} \equiv m_0 T_0^{-1} + r_h$, where T_0 and r_h are the string tension and the horizon radius respectively.

For B_2 to change its form from $\log(r)$ to behave as constant as $r \rightarrow \infty$, we will need three form flux that behaves as $1/r$ for small r to run as $1/r^n$, $n \geq 2$ for large r . To get such behavior, it is useful to define two functions $f(r)$ and $M(r)$ as (see Fig **2.18**):

$$f(r) \equiv \frac{e^{\alpha(r-r_0)}}{1 + e^{\alpha(r-r_0)}}, \quad M(r) \equiv M[1 - f(r)], \quad \alpha \gg 1 \quad (2.94)$$

where M is as before related to the effective number of five-branes (or the RR three-form charge) and r_0 is the location of the sources, which we will elaborate shortly. Note that for $r \ll r_0$, $f(r) \approx e^{r-r_0}$, whereas for $r > r_0$, $f(r) \approx 1$. Thus for r smaller than the scale r_0 , $f(r)$ is a very small quantity; whereas for r bigger than the scale r_0 , $f(r)$ is identity. In terms of $M(r)$ this means that for $r < r_0$, $M(r) \approx M$ whereas for $r > r_0$, $M(r) \rightarrow 0$. This will be useful below.

Using these functions, we see that one way in which logarithmic behavior along the radial direction can go to inverse r behavior, is when the warp factor takes the following form:

$$h = \frac{c_0 + c_1 f(r) + c_2 f^2(r)}{r^4} \sum_{\alpha} \frac{L_{\alpha}}{r^{\epsilon(\alpha)}} \quad (2.95)$$

where c_i are constant numbers, and the denominator can be mapped to $r_{(\alpha)}$ defined in (2.56) with $\epsilon_{(\alpha)}$ functions of $g_s N_f$, M , N and the resolution parameter a . L_{α} 's are functions of the angular coordinates (θ_i, ϕ_i, ψ) . For other details see [57]. The warp factor h has the required logarithmic behavior as long as the exponents of r are small and fractional numbers, and indeed switches to the inverse r behavior as soon as the exponents become integers. The question now is what are the background sources that give rise to a warp factor of the form (3.196). All the elements of OKS model only give rise to the logarithmic warp factor and it turns out, as we will discuss in some detail below, that we need to add sources (i.e. anti five branes) at the outermost

boundary of region 2 along with additional background fluxes. The specific point in the radial direction beyond which Region 3 starts tells us exactly *where* to add the sources and the AdS cap.

The demarcation point can be found by looking at the behavior of H_{NS} and H_{RR} . For this we need to use the functions (2.94) to write the RR three-form. Our ansatz for \tilde{F}_3 then is:

$$\begin{aligned} \tilde{F}_3 = & \left(a_o - \frac{3}{2\pi r g_s N_f} \right) \sum_{\alpha} \frac{2M(r)c_{\alpha}}{r^{\epsilon(\alpha)}} \left(\sin \theta_1 d\theta_1 \wedge d\phi_1 - \sum_{\alpha} \frac{f_{\alpha}}{r^{\epsilon(\alpha)}} \sin \theta_2 d\theta_2 \wedge d\phi_2 \right) \\ & \wedge \frac{e_{\psi}}{2} - \sum_{\alpha} \frac{3g_s M(r) N_f d_{\alpha}}{4\pi r^{\epsilon(\alpha)}} dr \wedge e_{\psi} \wedge \left(\cot \frac{\theta_2}{2} \sin \theta_2 d\phi_2 - \sum_{\alpha} \frac{g_{\alpha}}{r^{\epsilon(\alpha)}} \cot \frac{\theta_1}{2} \sin \theta_1 d\phi_1 \right) \\ & - \sum_{\alpha} \frac{3g_s M(r) N_f e_{\alpha}}{8\pi r^{\epsilon(\alpha)}} \sin \theta_1 \sin \theta_2 \left(\cot \frac{\theta_2}{2} d\theta_1 + \sum_{\alpha} \frac{h_{\alpha}}{r^{\epsilon(\alpha)}} \cot \frac{\theta_1}{2} d\theta_2 \right) \wedge d\phi_1 \wedge d\phi_2 \quad (2.96) \end{aligned}$$

where $a_o = 1 + \frac{3}{2\pi}$ and $(c_{\alpha}, \dots, h_{\alpha})$ are constants. One may also notice three things: first, how the internal forms get deformed near the innermost boundary of the region, second, how the function $f(r)$ appears for all the components, and finally, how N_f is not a constant but a delocalized function¹³. The function $f(r)$ becomes identity for $r > r_0$ and therefore $\tilde{F}_3 \rightarrow 0$ for $r > r_0$. For $r < r_0$, the corrections coming from $f(r)$ is exponentially small. Integrating \tilde{F}_3 over the topologically non-trivial three-cycle:

$$\frac{1}{2} e_{\psi} \wedge \left(\sin \theta_1 d\theta_1 \wedge d\phi_1 - \sum_{\alpha} \frac{f_{\alpha}}{r^{\epsilon(\alpha)}} \sin \theta_2 d\theta_2 \wedge d\phi_2 \right) \quad (2.97)$$

we find that the number of units of RR flux vary in the following way with respect to the radial coordinate r :

$$M_{\text{tot}}(r) = M(r) \left(1 + \frac{3}{2\pi} - \frac{3}{2\pi r g_s N_f} \right) \sum_{\alpha} \frac{c_{\alpha}}{r^{\epsilon(\alpha)}} \quad (2.98)$$

which is perfectly consistent with the RG flow, because for $r < r_0$, and $r \rightarrow r e^{-\frac{2\pi}{3g_s M}}$, M_{tot} decreases precisely as $M - N_f$ as the correction factor e^{r-r_0} coming from $f(r)$ is negligible. For $r > r_0$, M_{tot} shuts off completely. This also means that below r_0 , the total colors N decrease by M_{tot} exactly as one would have expected for the RG flow with N_f flavors. Using similar deformed internal forms, one can also write down the

¹³We will soon see that N_f in fact is the effective number of seven-branes.

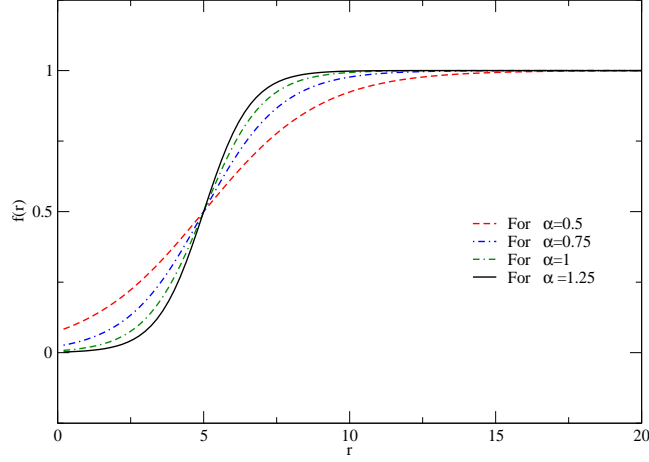


Figure 2.18: A plot of the $f(r)$ function for $r_0 = 5$ in appropriate units, and various choices of α . Observe that for large α the function quickly approaches 1 for $r > r_0$.

ansatz for the NS three-form. The result is:

$$\begin{aligned}
H_3 = & \sum_{\alpha} \frac{6g_s M(r) k_{\alpha}}{r^{\epsilon(\alpha)}} \left[1 + \frac{1}{2\pi} - \frac{\left(\text{cosec } \frac{\theta_1}{2} \text{ cosec } \frac{\theta_2}{2} \right)^{g_s N_f}}{2\pi r^{\frac{9g_s N_f}{2}}} \right] dr \\
& \wedge \frac{1}{2} \left(\sin \theta_1 d\theta_1 \wedge d\phi_1 - \sum_{\alpha} \frac{p_{\alpha}}{r^{\epsilon(\alpha)}} \sin \theta_2 d\theta_2 \wedge d\phi_2 \right) + \sum_{\alpha} \frac{3g_s^2 M(r) N_f l_{\alpha}}{8\pi r^{\epsilon(\alpha)}} \left(\frac{dr}{r} \wedge e_{\psi} - \frac{1}{2} de_{\psi} \right) \\
& \wedge \left(\cot \frac{\theta_2}{2} d\theta_2 - \sum_{\alpha} \frac{q_{\alpha}}{r^{\epsilon(\alpha)}} \cot \frac{\theta_1}{2} d\theta_1 \right) + g_s \frac{dM(r)}{dr} \left(b_1(r) \cot \frac{\theta_1}{2} d\theta_1 + b_2(r) \cot \frac{\theta_2}{2} d\theta_2 \right) \\
& \wedge e_{\psi} \wedge dr + \frac{3g_s}{4\pi} \frac{dM(r)}{dr} \left[\left(1 + g_s N_f - \frac{1}{r^{2g_s N_f}} + \frac{9a^2 g_s N_f}{r^2} \right) \log \left(\sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right) + b_3(r) \right] \\
& \sin \theta_1 d\theta_1 \wedge d\phi_1 \wedge dr - \frac{g_s}{12\pi} \frac{dM(r)}{dr} \left(2 - \frac{36a^2 g_s N_f}{r^2} + 9g_s N_f - \frac{1}{r^{16g_s N_f}} - \frac{1}{r^{2g_s N_f}} + \frac{9a^2 g_s N_f}{r^2} \right) \\
& \sin \theta_2 d\theta_2 \wedge d\phi_2 \wedge dr - \frac{g_s b_4(r)}{12\pi} \frac{dM(r)}{dr} \sin \theta_2 d\theta_2 \wedge d\phi_2 \wedge dr
\end{aligned} \tag{2.99}$$

with $(k_{\alpha}, \dots, q_{\alpha})$ being constants and $b_n = \sum_m \frac{a_{nm}}{r^{m+\epsilon_m}}$ where $a_{nm} \equiv a_{nm}(a^2, g_s N_f)$ and $\tilde{\epsilon}_m \equiv \tilde{\epsilon}_m(g_s N_f)$. The way we constructed the three-forms implies that H_3 is closed. In fact the $\mathcal{O}(\partial f)$ terms that we added to (2.99) ensures that. However F_3 is not closed. We can use the non-closure of F_3 to analyze *sources* that we need to add for consistency. These sources should in general be (p, q) five-branes, with (p, q)

negative, so that they could influence both the three-forms and since the Imaginary Self Duality (ISD) property of the three-forms is satisfied near $r = r_{\min}$ the sources should be close to the other boundary. A simplest choice could probably just be anti five-branes because adding anti D5-branes would change \tilde{F}_3 , and to preserve the ISD condition, H_3 would have to change accordingly. Furthermore, as we mentioned before, as $r \rightarrow r_0$, both $H_3 = \tilde{F}_3 \rightarrow 0$. Therefore $r = r_0$ is where Region 2 ends and Region 3 begins, and we can put the sources there. They could be oriented along the space time directions, located around the local neighborhood of $r = r_0$ and wrap the internal two-sphere (θ_1, ϕ_1) so that they are parallel to the seven-branes. However, putting in anti D5-branes near $r = r_0$ would imply non-trivial forces between the five-branes and seven-branes as well as five-branes themselves. Therefore if we keep, in general, the (p, q) five-branes close to say one of the seven-branes then they could get *dissolved* in the seven-brane as electric and magnetic gauge fluxes $*F^{(1)}$ and $F^{(1)}$ respectively. Here, as before, $*$ is the hodge star operator and $*F^{(1)}$ is the hodge dual $F^{(1)}$. In general a configuration of Dp branes is equivalent to a configuration of $Dp + 2$ brane with fluxes. This equivalence is due to M theory where a brane can be viewed as collapsed version of another higher dimensional brane. In type IIA theory, this was first shown by [73][74] where a $D0$ brane is considered as a tubular $D2$ brane with electric and magnetic fields on the $D2$ brane world volume. Thus we may treat the five brane charges being soaked in by seven-branes, which in turn would mean that \tilde{F}_3 in (2.96) and H_3 in (2.99) will satisfy the following EOMs:

$$\begin{aligned} d\tilde{F}_3 &= F^{(1)} \wedge \Delta_2(z) - d(\mathbf{Re} \tau) \wedge H_3 \\ d * H_3 &= * F^{(1)} \wedge \Delta_2(z) - d(C_4 \wedge F_3) \end{aligned} \quad (2.100)$$

where the tension of the seven-brane is absorbed in $\Delta_2(z)$, which is the term that measures the delocalization of the seven branes (for localized seven branes this would be copies of the two-dimensional delta functions) and τ is the axio-dilaton that we will determine below. In addition to that $d * F_3$ will satisfy its usual EOM.

However the above set of equations (2.100) is still not the full story. Due to the anti GSO projections between anti-D5 and D7-brane, there should be tachyon

between them. It turns out that the tachyon can be removed (or made massless) by switching on additional electric and magnetic fluxes on D7 along, say, (r, ψ) directions! This would at least kill the instability due to the tachyon, although susy may not be restored. For details on the precise mechanism on how brane-anti brane configurations are stabilized, one may refer to [73]-[78]. But switching on gauge fluxes on D7 would generate extra D5 charges and switching on gauge fluxes on anti-D5s will generate extra D3 charges. This is one reason why we write (N, N_f, M) as effective charges. This way a stable system of anti-D5s and D7 could be constructed.

To complete the rest of the story we need the axio-dilaton τ and the five-form. The five-form is easy to determine from the warp factor h using (2.41). The total five-form charge should have contribution from the gauge fluxes also, which in turn would affect the warp factor. For regions close to r_{\min} it is clear that τ goes as $z^{-g_s N_f}$ where z is the embedding (2.16). More generically and for the whole of Region 2, looking at the warp factor and the three-form fluxes, we expect the axio-dilaton to go as¹⁴:

$$\tau = [b_0 + b_1 f(r)] \sum_{\alpha} \frac{C_{\alpha}}{r^{\epsilon(\alpha)}} \quad (2.101)$$

where b_i are constants and C_{α} are functions of the internal coordinates and are complex. These C_{α} and the constants b_i are determined from the dilaton equation of motion [55, 56]:

$$\widetilde{\nabla}^2 \tau = \frac{\widetilde{\nabla} \tau \cdot \widetilde{\nabla} \tau}{i \text{Im } \tau} - \frac{4\kappa_{10}^2 (\text{Im } \tau)^2}{\sqrt{-g}} \frac{\delta S_{D7}}{\delta \tau} + (p, q) \text{ sources} \quad (2.102)$$

where tilde denote the unwarped internal metric g_{mn} , and S_{D7} is the action for the *delocalized* seven branes. The $f(r)$ term in the axio-dilaton come from the (p, q) sources that are absorbed as gauge fluxes on the seven-branes¹⁵. Because of this

¹⁴One may use this value of axio-dilaton and the three-form NS fluxes (2.99) to determine the beta function from the relations (2.21). To lowest order in $g_s N_f$ we will reproduce the beta functions presented in section 2.2.1. Notice that for $r > r_0$ the beta function *does not* vanish and both the gauge groups flow at the same rate. This will be crucial for our discussion in the following subsection.

¹⁵The $r^{-\epsilon(\alpha)}$ behavior stems from additional anti seven-branes that we need to add to the existing

behavior of axio-dilaton we don't expect the unwarped metric to remain Ricci-flat to the lowest order in $g_s N_f$. The Ricci tensor becomes:

$$\widetilde{\mathcal{R}}_{mn} = \kappa_{10}^2 \frac{\partial_{(m} \partial_{n)} \tau}{4(\text{Im } \tau)^2} + \kappa_{10}^2 \left(\widetilde{T}_{mn}^{\text{D7}} - \frac{1}{8} \widetilde{g}_{mn} \widetilde{T}^{\text{D7}} \right) + \kappa_{10}^2 \left(\widetilde{T}_{mn}^{(p,q)5\text{-brane}} - \frac{1}{4} \widetilde{g}_{mn} \widetilde{T}^{(p,q)5\text{-brane}} \right) \quad (2.103)$$

where we see that $\widetilde{\mathcal{R}}_{rr}$ picks up terms proportional to $\epsilon_{(\alpha)}^2$ and derivatives of $f(r)$, $N_f(r)$, implying that to zeroth order in $g_s N_f$ the interpolating region may not remain Ricci-flat. However since the coefficients are small, the deviation from Ricci-flatness is consequently small.

Finally the warp factor can be obtained using the five-form equation of motion:

$$d * dh^{-1} = H_3 \wedge \widetilde{F}_3 + \kappa_{10}^2 \text{tr} \left(F^{(1)} \wedge F^{(1)} - \mathcal{R} \wedge \mathcal{R} \right) \Delta_2(z) + \kappa_{10}^2 \text{tr} F^{(2)} \widetilde{\Delta}_4(\mathcal{S}) \quad (2.104)$$

where $F^{(1)}$ is the seven-brane gauge fields that we discussed earlier, $F^{(2)}$ is the (p, q) five-brane gauge fields required for stabilization of five brane anti-five brane configuration and they also give proper interpretation of the colors in the gauge theory side¹⁶. Here \mathcal{R} is the pull-back of the Riemann two-form, and $\widetilde{\Delta}_4(\mathcal{S})$ is the term that measures the delocalization of the dissolved (p, q) five-branes over the space \mathcal{S} embedded in the seven-brane (again for localized five-branes there would be copies of four-dimensional delta functions). The $H_3 \wedge \widetilde{F}_3$ term in (2.104) is proportional to $\frac{M^2(r)}{r^{2\epsilon(\alpha)}}$. This is precisely the form for the warp factor ansatz (3.196) with the $f^2(r)$ term there accounting for the $M^2(r)$ term above. This way with the warp factor (3.196) and the three-forms (2.96) and (2.99) we can satisfy (2.104) by switching on small gauge fluxes on the seven-branes and five-branes.

Therefore combining (3.196), (2.96), (2.99), (2.101) and the five-form, we can pretty much determine the supergravity background for the interpolating region

system to allow for the required UV behavior from the F-theory completion. The full picture will become clearer in the next sub-section when we analyze the system in Region 3.

¹⁶In fact one should view the gauge fluxes on the seven-branes and the five-branes as the total gauge fluxes that are needed to stabilize the system. We will see in the next subsection that the full stabilization would require additional fluxes, but the structure would remain the same.

$r_{\min} < r \leq r_0$. At the outermost boundary of Region 2 we therefore only have the metric and the axio-dilaton. Both the three-forms exponentially decay away fast, giving us a way to attach an AdS cap there.

2.4.3 Region 3: Seven Branes, F-Theory and UV Completions

The interpolating region, Region 2, that we derived above can be interpreted alternatively as the *deformation* of the neighboring geometry once we attach an AdS cap to the OKS-BH geometry. The OKS-BH geometry is the range $r_h \leq r \leq r_{\min}$ and the AdS cap is the range $r > r_0$. The geometry in the range $r_{\min} \leq r \leq r_0$ is the deformation. Such deformations should be expected for all other UV caps advocated in [57]. In this section we will complete the rest of the picture by elucidating the background from $r > r_0$ in the AdS cap. But before that let us give a brief gauge theory interpretation of background.

For the UV region $r > r_0$ we expect the dual gauge theory to be $SU(N + M) \times SU(N + M)$ with fundamental flavors coming from the seven-branes. This is because addition of (p, q) branes at the junction, or more appropriately anti five-branes at the junction with gauge fluxes on its world-volume, tell us that the number of three-branes degrees of freedom are $N + M$, with the M factor coming from five-branes anti-five-branes pairs. As mentioned in the previous section, one comes to this conclusion using M theory, as five anti-five brane configuration with flux is equivalent to three brane charge. Furthermore, the dual gauge theory for AdS conifold geometry is the Klenabov-Witten theory with gauge group $SU(\tilde{N}) \times SU(\tilde{N})$ for some \tilde{N} . Thus the gauge theory dual to region 3 is indeed $SU(N + M) \times SU(N + M)$, but has RG flows because of the fundamental flavors (This RG flow is the remnant of the flow that we saw in the previous subsection. We will determine this in more details below).

At the scale $r = r_0$ we expect one of the gauge groups to be Higgsed, so that we are left with $SU(N + M) \times SU(N)$. This Higgsing is justified as follows: the five and anti-five branes are separated by distance r_0 , which means the gauge bosons that

arise due to strings stretching between five and anti-five branes have length $\sim r_0$. Thus these bosons are massive with mass $\mathcal{O}(r_0 = \Lambda_0)$. For scales less than $r_0 = \Lambda_0$, these bosons are not produced and we only have the gauge theory arising from $D3, D5$ and seven branes i.e. OKS-BH type theory with gauge group $SU(N + M) \times SU(N)$. Now both the gauge fields flow at different rates and give rise to the cascade that is slowed down by the N_f flavors. In the end, at far IR, we expect confinement at zero temperature.

The few tests that we did above, namely, (a) the flow of N and M colors, (b) the RG flows, (c) the decay of the three-forms, and (d) the behavior of the dual gravity background, all point to the gauge theory interpretation that we gave above. What we haven't been able to demonstrate in [62] is the precise Higgsing that takes us from Klebanov-Witten type gauge theory to Klebanov-Strassler type cascading picture. From the gravity side its clear how this could be interpreted. From the gauge theory side it would be interesting to demonstrate this.

Coming back to the analysis of region 3, we see that in the region $r > r_0$ we do not expect three-forms but we do expect non-zero axio-dilaton. These non-zero axio-dilaton come from the the seven branes that are present in region 3. Of course the complete set of seven-branes should be determined from the F-theory picture [80][81] to capture the full non-perturbative corrections. This is now subtle because the seven-branes are embedded non-trivially here (see (2.16)). A two-dimensional base, parametrized by a complex coordinate z , on which we can have a torus fibration:

$$y^2 = x^3 + xF(z) + G(z) \quad (2.105)$$

can be identified with the z coordinate of (2.16). This way vanishing discriminant Δ of (2.105) i.e $\Delta \equiv 4F^3 + 27G^2 = 0$, will specify the positions of the seven-branes exactly as (2.16). Here we have taken $F(z)$ as a degree eight polynomial in z and $G(z)$ as a degree 12 polynomial in z .

As is well known, embedding of seven-branes in F-theory also tells us that we can have $SL(2, \mathbf{Z})$ jumps of the axio-dilaton [58][81]. We can define the axio-dilaton $\tau = C_0 + ie^{-\phi}$ as the modular parameter of a torus \mathbf{T}^2 fibered over the base parametrized

by the coordinate z . The holomorphic map¹⁷ from the fundamental domain of the torus to the complex plane is given by the famous j -function:

$$j(\tau) \equiv \frac{[\Theta_1^8(\tau) + \Theta_2^8(\tau) + \Theta_3^8(\tau)]^3}{\eta^{24}(\tau)} = \frac{4(24F(z))^3}{27G^2(z) + 4F^3(z)} \quad (2.106)$$

where $\Theta_i, i = 1, 2, 3$ are the well known Jacobi Theta-functions and η is the Dedekind η -function:

$$\eta(\tau) = q^{\frac{1}{24}} \prod_n (1 - q^n), \quad q = e^{2\pi i \tau} \quad (2.107)$$

For our purpose, we can write the discriminant $\Delta(z)$ and the polynomial $F(z)$ generically as:

$$\Delta(z) = 4F^3 + 27G^2 = a \prod_{j=1}^{24} (z - \tilde{z}_j), \quad F(z) = b \prod_{i=1}^8 (z - z_i) \quad (2.108)$$

where \tilde{z}_j are (like μ in (2.16)) complex constants and the j th seven brane is located at $z = \tilde{z}_j$. When we have weak type IIB coupling i.e $\tau = C_0 + i\infty$, $j(\tau) \approx e^{-2\pi i \tau}$ and using (2.106) the modular parameter can be mapped to the embedding coordinate z as:

$$\begin{aligned} \tau &= \frac{i}{g_s} + \frac{i}{2\pi} \log(55926ab^{-1}) - \frac{i}{2\pi} \sum_{n=1}^{\infty} \left[\frac{1}{nz^n} \left(\sum_{i=1}^8 3z_i^n - \sum_{j=1}^{24} \tilde{z}_j^n \right) \right] \\ &= \sum_{n=0}^{\infty} \frac{\mathcal{C}_n + i\mathcal{D}_n}{\tilde{r}^n} \end{aligned} \quad (2.109)$$

where $\mathcal{C}_n \equiv \mathcal{C}_n(\theta_i, \phi_i, \psi)$ and $\mathcal{D}_n \equiv \mathcal{D}_n(\theta_i, \phi_i, \psi)$ are real functions and $\tilde{r} = r^{3/2}$. To avoid cluttering of formulae, we will use r instead of \tilde{r} henceforth in this section unless mentioned otherwise. So the coordinate r will parameterize Region 3, and $\tau = \sum \frac{\mathcal{C}_n + i\mathcal{D}_n}{r^n}$.

The above computation was done assuming that $z > (z_i, \tilde{z}_j)$, which at this stage can be guaranteed if we take $\theta_{1,2}$ small. This gives rise to special set of configurations of seven-branes where they are distributed along other angular directions. However one might get a little worried if there exists some $\tilde{z}_j \equiv \tilde{z}_o$ related to the *farthest* seven-brane(s) where the above approximation fails to hold. This can potentially happen

¹⁷Holomorphic in τ , the modular parameter.

when we try to compute the mass of the heaviest quark in our theory. The question is whether we can still use the τ derived in (2.109), or we need to modify the whole picture.

Before we go into answering this question, the choice of z bigger than (z_i, \tilde{z}_j) already needs more convincing elaboration because allowing $\theta_{1,2}$ small is a rather naive argument. The situation at hand is more subtle than that and, as we will argue below, the picture that we have right now is incomplete.

To get the full picture, observe first that z being defined by equation (2.16) means that if we want (2.109) to hold in Region 3, we need to specify the condition $r > r_0$ in (2.16). This way a given z will always imply points in Region 3 for varying choices of the angular coordinates (θ_i, ϕ_i, ψ) . However a particular choice of (z_i, \tilde{z}_j) may imply very large r with small angular choices or small r with large angular choices. Thus analyzing the system only in terms of the r coordinate is tricky. In terms of the full complex coordinates, $z > (z_i, \tilde{z}_j)$ would mean that we are always looking at points away from the surfaces given by $z = z_i$ and $z = \tilde{z}_j$.

What happens when we touch the $z = z_i$ surfaces? For these cases $F(z_i) \rightarrow 0$ and therefore we are no longer in the weak coupling regime. For all $F(z_i) = 0$ imply $j(\tau) \rightarrow 0$ which in turn means $\tau = \exp(i\pi/3)$ on these surfaces. These are the constant coupling regimes of [82]-[84] where the string couplings on these surfaces are *not* weak. On the other hand, near any one of the seven-branes $z = \tilde{z}_j$ we are in the weak coupling regimes and so (2.109) will imply

$$\tau(z) = \frac{1}{2\pi i} \log(z - \tilde{z}_j) \rightarrow i\infty \quad (2.110)$$

which of course is expected but nevertheless problematic for us. This is because we need logarithmic behavior of axio-dilaton in Region 2, but not in Region 3. For a good UV behavior, we need axio-dilaton to behave like (2.109) everywhere in Region 3.

In addition to that there is also the issue of the heaviest quarks creating additional log divergences that we mentioned earlier. These seven branes are located at $z = \tilde{z}_j \equiv \tilde{z}_o$, and therefore if we can make the axio-dilaton independent of the coordinates \tilde{z}_o

then at least we won't get any divergences from these seven-branes. It turns out that there are configurations (or rearrangements) of seven-brane(s) that allow us to do exactly that. To see one such configuration, let us define $F(z)$, $G(z)$ and $\Delta(z)$ in (2.108) in the following way:

$$\begin{aligned} F(z) &= (z - \tilde{z}_o) \prod_{i=1}^7 (z - z_i), & G(z) &= (z - \tilde{z}_o)^2 \prod_{i=1}^{10} (z - \hat{z}_i) \\ \Delta(z) &= (z - \tilde{z}_o)^3 \prod_{j=1}^{21} (z - \tilde{z}_j) \end{aligned} \quad (2.111)$$

which means that we are stacking a bunch of *three* seven-branes at the point $z = \tilde{z}_o$, and

$$\prod_{j=1}^{21} (z - \tilde{z}_j) \equiv 4 \prod_{i=1}^7 (z - z_i)^3 + 27(z - \tilde{z}_o) \prod_{i=1}^{10} (z - \hat{z}_i)^2 \quad (2.112)$$

implying that the axio-dilaton τ becomes independent of \tilde{z}_o and behaves exactly as in (2.109) with (i, j) in (2.109) varying up to $(7, 21)$ respectively.

The situation is now getting better. We have managed to control a subset of log divergences. To get rid of the other set of log divergences that appear on the remaining twenty-one surfaces, one possible way would be to modify the embedding (2.16). In fact a change in the embedding equation will also explain the axio-dilaton choice (2.101) of Region 2. To change the embedding equation (2.16) we will use similar trick that we used to kill off the three-form fluxes, namely, attach anti-branes. These anti seven-branes¹⁸ are embedded via the following equation:

$$r^{3/2} e^{i(\psi - \phi_1 - \phi_2)} \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} = r_0 e^{i\Theta} \quad (2.113)$$

where Θ is some angular parameter, and could vary for different anti seven-branes. The above embedding will imply that their overlaps with the corresponding seven-branes are only partial¹⁹. And since we require

$$|\tilde{z}_j|^{2/3} < r_o$$

¹⁸They involve both local and non-local anti seven-branes.

¹⁹For example if we have a seven-brane at $z = \tilde{z}_1$ such that lowest point of the seven brane is $r = |\tilde{z}_1|^{2/3} < r_o$, then the corresponding anti-brane has only partial overlap with this.

it will appear effectively that we can only have seven-branes in Regions 1 and 2, and *bound* states of seven-branes and anti seven-branes in Region 3.²⁰ This way the axio-dilaton in Region 3 will indeed behave as (2.109), with the seven values of z_i in (2.112) chosen to be in region 1 and 2.

There are two loose ends that we need to tie up to complete this side of the story. The first one is the issue of Gauss' law, or more appropriately, charge conservation. The original configuration of 24 seven branes had zero global charge, but now with the addition of anti seven-branes charge conservation seems to be problematic. There are a few ways to resolve this issue. First, we can assume that that branes wrap topologically trivial cycles, much like the ones of [38]. Then charge conservation is automatic. The second alternative is to isolate six seven-branes using some appropriate F and G functions, so that they are charge neutral. This is of course one part of the constant coupling scenario of [58]. Now if we make the (θ_2, ϕ_2) directions non-compact then we can put in a configuration of 18 seven-branes and anti seven-branes pairs together using the embeddings (2.16) and (2.113) respectively. The system would look effectively like what we discussed above. Since the whole system is now charge neutral, compactification shouldn't be an issue here.

The second loose end is the issue of tachyons between the seven-brane and anti seven-brane pairs. Again, as for the anti-D5 branes and D7-brane case [73]-[78], switching on appropriate electric and magnetic fluxes will make the tachyon massless! Therefore the system will be stable and would behave exactly as we wanted, namely, the axio-dilaton will not have the log divergences over any slices in Region 3.

This behavior of axio-dilaton justifies the $r^{-\epsilon(\alpha)}$ in (2.101) in Region 2. So the full picture would be a set of seven-branes with electric and magnetic fluxes embedded via (2.16) and another set of anti seven-branes embedded via (2.113) lying completely in Region 3.

Thus in Region 3 both the three-forms vanish and therefore $g_1 = g_2 = g_{\text{YM}}$ with

²⁰Of course this effective description is only in terms of the axio-dilaton charges. In terms of the embedding equation for the seven-branes (2.16) this would imply that we can define z with $r > r_o$ and \tilde{z}_j with $r < r_o$.

g_1, g_2 being the couplings for $SU(N+M), SU(N+M)$. From (2.21) we can compute the β -function for g_{YM} as:

$$\beta(g_{\text{YM}}) \equiv \frac{\partial g_{\text{YM}}}{\partial \log \Lambda} = \frac{g_{\text{YM}}^3}{16\pi} \sum_{n=1}^{\infty} \frac{n \mathcal{D}_n}{\Lambda^n} \quad (2.114)$$

where Λ is the usual RG scale related to the radial coordinate in the supergravity approximation. For $\Lambda \rightarrow \infty$, $\beta(g_{\text{YM}}) \rightarrow 0$ implying a conformal theory in the far UV. We can fix the 't Hooft coupling to be strong to allow for the supergravity approximation to hold consistently at least for all points away from the $z = z_i, i = 1, \dots, 7$ surfaces.

Existence of axio-dilaton τ of the form (2.109) and the seven-brane sources will tell us, from (2.103), that the unwarped metric may not remain Ricci flat. For example it is easy to see that

$$\widetilde{\mathcal{R}}_{rr} = \frac{\mathcal{A}_{\mathcal{D}}}{r^2 \mathcal{D}_0^2} \sum_{n,m=1}^{\infty} nm \frac{(\mathcal{C}_n + i\mathcal{D}_n)(\mathcal{C}_m - i\mathcal{D}_m)}{r^{n+m}} + \mathcal{O}\left(\frac{1}{r^n}\right) \quad (2.115)$$

where the last term should come from the seven-brane sources and, because of these sources, we don't expect $\widetilde{\mathcal{R}}_{rr}$ to vanish to lowest order in $g_s N_f$.²¹ The term $\mathcal{A}_{\mathcal{D}}$ is given by the following infinite series:

$$\mathcal{A}_{\mathcal{D}} = 1 - \sum_{k,l=1}^{\infty} \frac{\mathcal{D}_k \mathcal{D}_l \mathcal{D}_0^{-2}}{r^{k+l}} + \sum_{k,l,p,q=1}^{\infty} \frac{\mathcal{D}_k \mathcal{D}_l \mathcal{D}_p \mathcal{D}_q \mathcal{D}_0^{-2}}{r^{k+l+p+q}} + \dots \quad (2.116)$$

Similarly one can show that

$$\widetilde{\mathcal{R}}_{ab} = \frac{\mathcal{A}_{\mathcal{D}}}{\mathcal{D}_0^2} \sum_{n,m=0}^{\infty} \frac{(\partial_a \mathcal{C}_n + i\partial_a \mathcal{D}_n)(\partial_b \mathcal{C}_m - i\partial_b \mathcal{D}_m)}{r^{n+m}} + \mathcal{O}\left(\frac{1}{r^n}\right) \quad (2.117)$$

for $(a, b) \neq r$. For $\widetilde{\mathcal{R}}_{rb}$ similar inverse r dependence can be worked out. In the far UV we expect the unwarped curvatures should be equal to the AdS curvatures. The warp factor h on the other hand can be determined from the following variant of (2.104):

$$d * dh^{-1} = \kappa_{10}^2 \text{tr} \left(F^{(1)} \wedge F^{(1)} - \mathcal{R} \wedge \mathcal{R} \right) \Delta_2(z) + \dots \quad (2.118)$$

because we expect no non-zero three-forms in Region 3. The dotted terms are the non-abelian corrections from the seven-branes. As r is increased i.e $r \gg r_0$, we

²¹Although, as discussed before, the deviation from Ricci flatness will be very small.

expect $F^{(1)}$ to fall-off (recall that they appear from the anti (1,1) five-branes located in the neighborhood of $r = r_0$) and therefore can be absorbed in \mathcal{R} . Once we embed the seven-brane gauge connection in some part of spin-connection, we expect

$$\square h^{-1} = \mathcal{O}\left(\frac{1}{r^n}\right) \quad (2.119)$$

where the box represents combinations of differential operators that arise from (2.118). Solving this will reproduce the generic form for h :

$$h = \frac{L^4}{r^4} \left[1 + \sum_{i=1}^{\infty} \frac{a_i(\psi, \theta_i, \phi_i)}{r^i} \right] \quad (2.120)$$

with a constant L^4 and a_i 's are suppressed by powers of $g_s N_f$. More details on this is given in the Appendix A and B of [62]. At far UV we recover the AdS picture implying a strongly coupled conformal behavior in the dual gauge theory.

To summarize, we have obtained the dual gravity of a thermal field theory with matter in the fundamental representation. The gauge theory becomes almost conformal in the UV with massive Higgs-like gauge bosons with matter transforming under $SU(N + M) \times SU(N + M)$ gauge symmetry. In the IR the massive gauge bosons are Higgsed away and we end up with a $SU(N + M) \times SU(N)$ gauge theory. The dual geometry has the metric of the form (2.91) with a warp factor given by (2.31) for $r \ll r_0$, by (3.196) for $r \sim r_0$, and of the form (2.120) for $r \gg r_0$. Thus in UV we have Klebanov-Witten type field theory while in the IR we have Klebanov-Strassler model with fundamental matter and temperature. Once we reach the Klebanov-Strassler type description of our field theory, we expect Seiberg duality cascade to occur as we go further down in energy scale. At the bottom of the cascade we end up with strongly coupled $SU(\bar{M})$ gauge theory with N_f fundamental flavors and logarithmic running of coupling. The scenario is sketched in Fig 2.19.

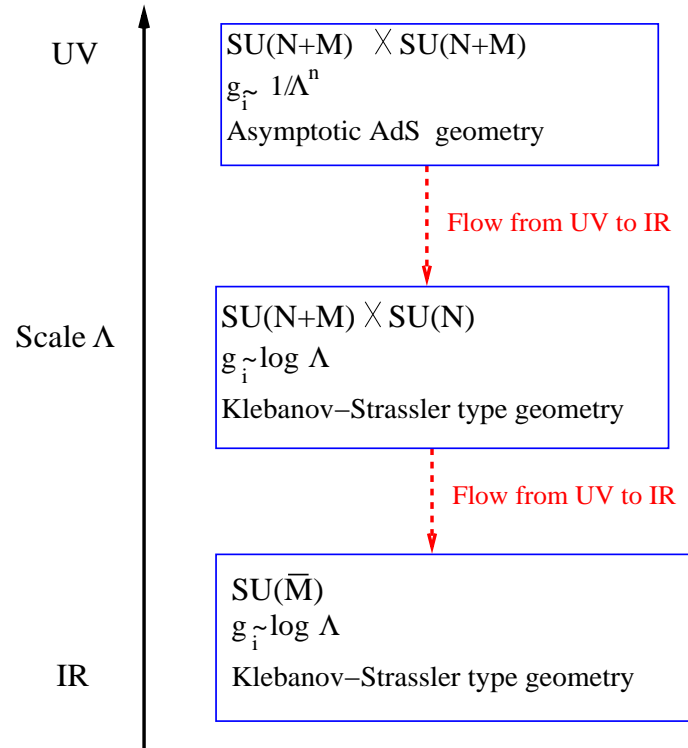


Figure 2.19: Schematic depiction of flow in the gauge theory

Chapter 3

Application to Thermal QCD

Having discussed in some detail the gauge theories that arise from various brane setups and their dual geometries, now we attempt to make connections with thermal QCD in the large N limit. The gauge theory arising from the brane setup in section 2.4 is obtained by UV completing OKS-BH geometry and is asymptotically conformal with logarithmic running of couplings in the IR. To our knowledge it is the brane setup that resembles large N QCD the most in the ‘top down’ approach where gauge/gravity duality is exactly derivable. Unlike the ‘bottom up’ approach where we do not know the exact brane configuration in higher dimensions or the precise origin of the gauge theory, our approach is based on open/closed string duality and in some sense the correspondence is more theoretically complete.

As sketched in Fig 2.19, although the gauge theory has a very rich structure, it is of course not exactly QCD. However, the benefit of this construction is at large coupling it has an exact classical supergravity description. Certain gauge theory observables which are extremely difficult to obtain using conventional field theory techniques become rather simple to calculate using dual gravity. The field theory lives in four flat spatial dimensions and incorporates matter in fundamental representation. We will refer to the fundamental matter as ‘quarks’ of our theory. The theory has no UV Landau poles while effective degrees of freedom converge in the far UV. As the theory reaches conformal fixed point at large energy scales, our construction maybe viewed as the UV completion of Klebanov-Strassler model which is most relevant for QCD.

Although the field content is somewhat different from large N QCD, our theory has several common features to it and in fact becomes almost large N QCD in the IR. Thus the physics extracted from our gauge theory could be very similar to that of QCD and it is worth analyzing the thermodynamic properties of the field theory plasma. As this may turn out to be crucial in understanding the collective excitations of QCD plasma, in this chapter we will examine the dynamics of ‘quarks’, transport coefficients of the medium and the confining nature of the our gauge theory.

3.1 Quark Dynamics

Strings in ten dimensional dual gravity with endpoint on a D7 brane corresponds to fundamental matter i.e. quarks in four dimensional Minkowski space time. As the D7 brane fills the four dimensional Minkowski space, the endpoint of the string is a point there and the quark is localized at the endpoint. Energy of the string gives energy of the quark and minimal energy of the static string configuration in the bulk geometry gives the mass of the quark. If we consider a string traveling through the bulk geometry, it corresponds to a quark moving through gauge theory plasma. In the following subsections we will use this key concept to compute thermal mass of a quark and the drag it experiences as it traverses a medium. We will also quantify the wake it leaves behind in the medium along with transverse momentum broadening of a fast moving quark.

3.1.1 Thermal Mass and Drag of a Quark

We start with the action of a string in ten dimensional geometry with metric (2.23). If $X^i(\sigma, \tau)$ is a map from world sheet coordinates σ, τ to 10 dimensional space time, then string action or fundamental string Born Infeld action is (see for example [85][86]):

$$S_{\text{string}} = T_0 \int d\sigma d\tau \left[\sqrt{-\det(f_{\alpha\beta} + \partial_\alpha \phi \partial_\beta \phi)} + \frac{1}{2} \epsilon^{ab} B_{ab} + J(\phi) \right. \\ \left. + \partial X^m \partial X^n \bar{\Theta} \Gamma_m \Gamma^{abc\dots} \Gamma_n \Theta F_{abc\dots} + \mathcal{O}(\Theta^4) \right] = \int d^{10}x \mathcal{L}_{\text{string}}(x)$$

where $J(\phi)$ is the additional coupling of the dilaton ϕ to the string world-sheet, T_0 is the string tension, X^n are the ten bosonic coordinates, Θ is a 32 component spinor, $F_{abc\dots} = [dC]_{abc\dots}$ with $C_{abc\dots}$ being the background RR form potentials, and $f_{\gamma\delta}$ is world sheet metric, given by the standard pull-back of the space time metric on the world-sheet:

$$f = \begin{pmatrix} \dot{X} \cdot \dot{X} & \dot{X} \cdot X' \\ \dot{X} \cdot X' & X' \cdot X' \end{pmatrix} = \begin{pmatrix} \frac{\dot{X}^2}{\sqrt{h}} - \frac{g_1}{\sqrt{h}} & \frac{\dot{X}X'}{\sqrt{h}} \\ \frac{\dot{X}X'}{\sqrt{h}} & \frac{\sqrt{h}}{g_2} + \frac{X'^2}{\sqrt{h}} \end{pmatrix}$$

where we have taken the background metric of the form (2.91) ignoring $g_s^2 N_f^2$ terms which means g_{mn} is the metric of $T^{1,1}$ in (2.91). Here $\gamma, \delta = 0, 1$ with parametrization $\eta^0 = \tau = t$ and $\eta^1 = \sigma = r$.

In the ensuing analysis we will keep B_{ab} , $J(\phi)$ as well as $\partial_a \phi$ zero and the justification will be given shortly. The interesting thing however is to do with the background RR forms. Note that the RR forms *always* couple to the 32 component spinor. Therefore once we switch-off the fermionic parts in (3.1), the fundamental string is completely unaffected by the background RR forms¹. Thus in the following analysis, for the mass and drag of the quark, we can safely ignore the RR fields. We have also defined:

$$\begin{aligned} \det f &= -G_{ij} \dot{X}^i X'^j + (G_{ij} X'^i X'^j)(G_{kl} \dot{X}^k \dot{X}^l) \\ X'^i &= \frac{\partial X^i}{\partial \sigma}, \quad \dot{X}^i = \frac{\partial X^i}{\partial \tau} \end{aligned} \tag{3.1}$$

where G_{ij} is more generic than the background metric, and could involve the back reaction of the fundamental string on the geometry. The analysis is very similar to the AdS case discussed in [87][88], however, since our background geometry in [57] involves running couplings, the results will differ from the ones of [87][88].

As mentioned earlier, a fundamental quark will be a string starting from the D7 brane and ending on a D3 or fractional D3 (which is a D5 brane) brane, giving the quark color. At non zero temperature, the dual geometry has a black hole and quark

¹This is of course the familiar statement that the RR fields do not couple in a simple way to the fundamental string.

in the thermal medium is represented by a string stretching between seven brane and the horizon r_h of the black hole. For simplicity of the calculation, we will then restrict to the case when

$$\begin{aligned} X^0 = t, \quad X^4 = r, \quad X^1 = x(\sigma, \tau), \quad X^k = 0 \quad (k = 2, 3, 5, 6, 7) \\ (X^8, X^9) = (\theta_1, \theta_2) = \pi, \quad \Theta = \bar{\Theta} = 0 \end{aligned} \quad (3.2)$$

and we choose parametrization $\tau = t, \sigma = r$ also known as the static gauge. Thus we are only considering the case when the string extends in the r direction, does not interact with the RR fields, and moves in the x direction of our manifold. More general string profile, while being computationally challenging, does not introduce any new physics and hence our simplification is a reasonable one.

Before moving further, let us consider two points. First is the effect of the black hole on the *shape* of the D7 brane. We expect due to gravitational effects the D7 brane will sag towards the black hole and eventually the string would come very close to the horizon. In fact putting a point charge on the D-brane tends to create a long thin tube on the D-brane that in general extends to infinity. The end point of the string being a source of point charge should show similar effects (see [90] for a discussion of a somewhat similar scenario)². For our analysis here we will ignore this effect altogether and we hope to address the issue in our future work.

The second point is to see how the background varying dilaton and NS-NS two form effect the string. With the axion and dilaton behaving as (2.41) near the location of D7 brane and as $\mathcal{O}(g_s N_f / r^n)$ globally, $\partial_\alpha \phi \sim \mathcal{O}(g_s N_f / r^m)$ for all r at the location of the string. Note that $g_s N_f \ll 1$ and as the string extends from $r = r_0$ to $r = r_h$, for r_h large i.e. high temperature, $\mathcal{O}(g_s N_f / r^m)$ is negligible at the location of the string. This means we can ignore the contribution from $\partial_\alpha \phi$ in the string action. Similarly we can ignore effects of B_2 as it gives $\mathcal{O}(g_s M \log r)$ for small r and $\mathcal{O}(g_s M / r^m)$ for large r , both of which can be ignored at the location of the string. This is because

²Even in the supersymmetric case, putting a point charge on a D-brane tends to create a long tube that extends to infinity. One can then view an open string to lie at the end of the thin tube. This effect is somewhat similar to the one discussed in [89].

for reasonably large temperature, the IR divergence of $\log(r)$ is absent (temperature which is of $\mathcal{O}(r_h)$ is the IR cutoff) and at the UV we can ignore $1/r^n$ terms. On the other hand, the dilaton additionally couples to the string world sheet through $J(\phi)$ term which is proportional to the Ricci scalar $R_{(2)}$ of the world sheet metric, $f_{\gamma\delta}$ and under a reparametrization of the world sheet metric, we can make $R_{(2)}$ small enough as there are no divergences in the background metric. This way we can ignore $J(\phi)$ near the location of the string. However in section **3.3** we will consider the effects of B_2 and dilaton on the string world sheet and we will see that these effects although rather technical to track, do not change the overall physics. Thus we can simply set $J(\phi) = B_2 = \partial_\mu \phi = 0$ in the following analysis.

Getting back to the string action, with our choice of parametrization and string profile we get:

$$-\det f = \frac{g_1(r)}{g_2(r)} + \frac{g_1(r)}{h(r, \pi, \pi)} x'^2 - g_1(r)^{-1} \dot{x}^2 \quad (3.3)$$

where the warp factor $h(r, \theta_1, \theta_2) = h(r, \pi, \pi)$, evaluated at the location of the string. With this, the rest of the analysis is a straightforward extension of [87][88]. The Euler-Lagrangian equation for $X^1 = x(t, r)$ derived from the action (3.1) and the associated canonical momenta are:

$$\begin{aligned} \frac{1}{g_2} \frac{d}{dt} \left(\frac{\dot{x}}{\sqrt{-\det f}} \right) + \frac{d}{dr} \left(\frac{g_1 x'}{h \sqrt{-\det f}} \right) &= 0 \\ \Pi_i^0 &= -T_0 G_{ij} \frac{(\dot{X} \cdot X')(X^j)' - (X')^2 (\dot{X}^j)}{\sqrt{-\det f}} \\ \Pi_i^1 &= -T_0 G_{ij} \frac{(\dot{X} \cdot X')(\dot{X}^j) - (\dot{X})^2 (X^j)'}{\sqrt{-\det f}} \end{aligned} \quad (3.4)$$

If we consider a static string configuration, i.e. $x(\sigma, \tau) = b = \text{constant}$, then energy can be interpreted as the thermal mass of the quark in the dual gauge theory. Using the static solution in (3.4), we obtain the thermal mass $m(\mathcal{T})$ using $E = -\int d\sigma \Pi_t^0$, as

$$m(\mathcal{T}) = T_0(r_0 - r_h) = T_0 \left(|\mu|^{2/3} - \mathcal{T} \right) \quad (3.5)$$

Observe that the thermal mass decreases as temperature is increased. In our analysis, $\mu^{2/3} = r_0 > \mathcal{T}$ and $\mu^{2/3}$ is proportional to zero temperature mass. This

means, we are dealing with heavy quarks where the temperature is always less than the mass of the quark. On the other hand, our predictions for the thermal mass gets exact in the limit where number of colors is infinite. Thus our results suggest that at large N and small temperatures, the heaviest quarks get less massive as temperature is increased.

When a probe particle moves through a plasma, it interacts with the medium through collisions with the constituents of the medium and if it is charged, it also radiates. Overall because of the interaction with the medium, the probe experiences drag force and this drag is a key characteristic of the plasma. As a moving string in the bulk corresponds to a moving quark in the medium, we can compute the drag coefficient by considering a string moving with some velocity with the endpoint representing a quark which traverses the medium. For the computation of drag, that is to analyze the response of the medium to a moving probe, it is enough to consider a constant velocity quark and calculate the drag it experiences. Of course drag will try to slow down the quark and we need to apply force to keep it at constant speed. If there is no external force applied, the probe will slow down but the drag coefficient we extract considering constant speed remains unchanged - as we can consider instantaneous velocity to be the constant velocity and the analysis stays the same.

We consider the string profile (3.2) with

$$x(t, r) = \bar{x}(r) + vt \quad (3.6)$$

Then from (3.4), noting that f is independent of time, we can solve the equation of motion to get:

$$\bar{x}^{\prime 2} = \frac{h^2 C^2 v^2}{g_1 g_2} \cdot \frac{g_1 - v^2}{g_1 - h C^2 v^2} \quad (3.7)$$

where C is a constant of integration that can be determined by demanding that $-\det f$ is always positive. Using the value of $\bar{x}^{\prime 2}$ from (3.7) we can give an explicit expression for the determinant of f as:

$$-\det f = \frac{g_1}{g_2} \cdot \frac{g_1 - v^2}{g_1 - h C^2 v^2} \quad (3.8)$$

For $-\det f$ to remain positive for all r , we need both numerator and denominator to change sign at same value of r . This is the same argument as in [87] [88]. The numerator changes sign at³

$$r^2 = \frac{r_h^2}{\sqrt{1-v^2}} + \mathcal{O}(g_s N_f, g_s M) \quad (3.9)$$

where we use h near horizon to be of the form in (2.31).⁴ Requiring that denominator also change sign at that value fixes C to be:

$$C = \frac{r_h^2 L^{-2}}{\sqrt{1-v^2}} \cdot \frac{1}{\sqrt{1 + \frac{3g_s \bar{M}^2}{2\pi N} \log\left[\frac{r_h}{(1-v^2)^{1/4}}\right] \left(1 + \frac{3g_s \bar{N}_f}{2\pi} \left\{\log\left[\frac{r_h}{(1-v^2)^{1/4}}\right] + \frac{1}{2}\right\}\right)}} \quad (3.10)$$

where \bar{M} and \bar{N}_f differs from M, N_f due to the $\mathcal{O}(g_s N_f, g_s M)$ terms in (3.9). The first part of C is the one derived in [87] [88]. The next part is new. Now the rate at which momentum is lost to the black hole is given by the momentum density at horizon

$$\Pi_1^x(r = r_h) = -T_0 C v \quad (3.11)$$

while the force quark experiences due to friction with the plasma is $\frac{dp}{dt} = -\nu p$ with $p = mv/\sqrt{1-v^2}$. To keep the quark moving at constant velocity, an external field \mathcal{E}_i does work and the equivalent energy is dumped into the medium [87][88]. Thus the rate at which a quark dumps energy and momentum into the thermal medium is precisely the rate at which the string loses energy and momentum to the black hole. Thus up to $\mathcal{O}(g_s N_f, g_s M)$ we have $\frac{\nu mv}{\sqrt{1-v^2}} = -\Pi_1^x(r = r_h)$ and

$$\begin{aligned} \nu &= \frac{T_0 C \sqrt{1-v^2}}{m} \\ &= \frac{T_0}{m L^2} \frac{\mathcal{T}^2}{\sqrt{1 + \frac{3g_s \bar{M}^2}{2\pi N} \log\left[\frac{\mathcal{T}}{(1-v^2)^{1/4}}\right] \left(1 + \frac{3g_s \bar{N}_f}{2\pi} \left\{\log\left[\frac{\mathcal{T}}{(1-v^2)^{1/4}}\right] + \frac{1}{2}\right\}\right)}} \end{aligned} \quad (3.12)$$

³Note that by $\mathcal{O}(g_s N_f, g_s M)$ we will always mean $\mathcal{O}(g_s N_f, g_s^2 M N_f, g_s M^2/N)$ unless mentioned otherwise.

⁴This is justified as for all the geometries considered in section **2.2-2.4**, the near horizon $r \sim r_h$ warp factor is always of the form (2.31). We have various choices for the large r behavior of the geometry and subsequently for h , but the IR behavior remains the same as the OKS model.

which should now be compared with the AdS result [87][88]. In the AdS case the drag coefficient ν is proportional to \mathcal{T}^2 . For our case, when we incorporate RG flow in the gravity dual, we obtain $\mathcal{O}\left(1/\sqrt{A \log \mathcal{T} + B \log^2 \mathcal{T}}\right)$ correction to the drag coefficient computed using AdS/CFT correspondence [87] [88].

3.1.2 Transverse Momentum Broadening

With the computation of thermal mass of the quark and the drag it experiences in the medium, we will now study the diffusion process through which a probe particle transfers momentum with the plasma. The analysis is of particular interest as it relates to the formation of the quark gluon plasma at the relativistic heavy ion colliders. To be more precise, at the earliest stages of a central collision, energy densities are expected to be high enough to form the quark gluon plasma and several observables have been proposed to probe the plasma. Strangeness enhancement [91], J/ψ suppression [92], electromagnetic radiation [93] are among the key candidates while the ones that are produced at the earliest stages of the collision gather special attention as they directly interact with the plasma. Hard scatterings where the partons transfer large momentum take place right after the collision as only then there is enough energy and the resulting partons fragment into jets of hadrons with large transverse momentum P_T . The partons produced in the heavy ion collisions are expected to lose energy in the medium [94] which should result in a suppression of high P_T hadrons [95] when compared to the high P_T hadrons produced in proton-proton collision. Experiments at RHIC have indeed observed this suppression [96]-[98] and this phenomenon is known as ‘jet quenching’.

There has been a lot of effort to model the energy loss mechanism [99]-[109] and gluon bremsstrahlung with Landau-Pomeranchuk-Migdal (LPM) effect [110] is thought to be the dominant process in jet quenching. As the medium formed in the collision expands, one needs to model the time evolution of the system accounting the dynamics of the fluid and then compare with experimental data of suppression. As the dual geometry we constructed is time independent, we cannot address the effects due to the evolution of medium. Rather we will quantify the diffusion process by

computing the mean square transverse momentum transfer between the *non expanding* medium and a fast moving parton. In principle, one needs to construct a time dependent dual geometry by considering the collision of strings ending on D7 branes and computing their back reaction, similar in the spirit of [111]. Then using this time dependent background geometry to compute Wilson loops, one can calculate momentum distribution and finally quantify ‘jet quenching’. Constructing the dual geometry of a heavy ion collision is rather challenging and we hope to address the issue in our future work.

For our current analysis, consider a parton moving through a plasma in four dimensional Minkowski space time with the following world line

$$\begin{aligned} x(t) &= vt \\ z(t) &= y(t) \equiv \delta y(t) \end{aligned} \quad (3.13)$$

For a fast moving parton, we can always choose coordinates such that (3.13) is the world line. Now if the plasma has matter in fundamental representation, has logarithmic running of coupling in the IR but becomes asymptotically conformal, we can treat the ten dimensional geometry with metric of the form (2.24) to be the dual gravity of this gauge theory which lives in four dimensional flat space time.

To obtain the momentum broadening of the parton we shall use the Wigner distribution function f as defined in QCD kinetic theory [112]

$$f(X, r_\perp) \equiv \langle f_{cc}(X, r_\perp) \rangle = \text{Tr} \left[\rho Q_a^\dagger(X_-^\perp) U_{ab}(X_-^\perp, X_+^\perp) Q_a(X_+^\perp) \right] \quad (3.14)$$

where $X_\pm^\perp = X - r_\perp/2$, $X_\pm^\perp = X + r_\perp/2$, $X = (t, \mathbf{x})$ is the world line of the parton field Q_a without any fluctuation, U_{ab} is the link and ρ is the density matrix. Of course the index ab refers to color and for our choice of the world line for the parton we have $r_\perp^2 = 2\delta y^2$. Now Fourier transforming the distribution functions, we can get the average transverse momentum to be

$$\langle p_\perp^2 \rangle = \int d^3x \int \frac{d^2k_\perp}{(2\pi)^2} k_\perp^2 f(X, k_\perp) \quad (3.15)$$

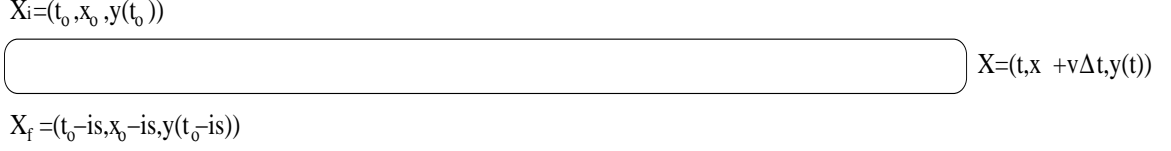


Figure 3.1: The contour C with X denoting the coordinates of a point on the contour. The upper line with real time coordinate is the world line of the heavy parton while the lower line has complex time coordinate. Here $s \rightarrow 0$ is real.

Now if we assume that the initial transverse momentum distribution is narrow, then we have

$$\langle p_{\perp}^2 \rangle = 2\kappa_T \mathcal{T} \quad (3.16)$$

with \mathcal{T} some large time interval, and κ_T is the diffusion coefficients.

Following the arguments in [114], one can write the diffusion coefficients solely in terms of functional derivative of Wilson loops. The final result is

$$\kappa_T = \lim_{\omega \rightarrow 0} \frac{1}{4} \int dt e^{i\omega t} (iG_{11}^y(t, 0) + iG_{22}^y(t, 0) + iG_{12}^y(t, 0) + iG_{21}^y(t, 0)) \quad (3.17)$$

where the Greens functions are

$$\begin{aligned} G_{11}^y(t, t') &= \frac{1}{\text{tr}\rho^0 W_C[0, 0]} \langle \text{tr}\rho^0 \frac{\delta^2 W_C[\delta y_1, 0]}{\delta y_1(t) \delta y_1(t')} \rangle \\ G_{22}^y(t, t') &= \frac{1}{\text{tr}\rho^0 W_C[0, 0]} \langle \text{tr}\rho^0 \frac{\delta^2 W_C[0, \delta y_2]}{\delta y_2(t) \delta y_2(t')} \rangle \\ G_{12}^y(t, t') &= \frac{1}{\text{tr}\rho^0 W_C[0, 0]} \langle \text{tr}\rho^0 \frac{\delta^2 W_C[\delta y_1, \delta y_2]}{\delta y_1(t) \delta y_2(t')} \rangle \\ G_{21}^y(t, t') &= \frac{1}{\text{tr}\rho^0 W_C[0, 0]} \langle \text{tr}\rho^0 \frac{\delta^2 W_C[y_2(t), \zeta_2(t')]}{\delta y_1(t) \delta y_2(t')} \rangle \end{aligned} \quad (3.18)$$

where t, t' are the real part of complex time t_C, t'_C on contour C (Fig **3.1**). Here we have introduced type ‘1’ and ‘2’ fields ($\delta y_i, i = 1, 2$) of thermal field theory in

real time formalism [115][116] evaluated with real and complex time coordinates - hence evaluated at the upper and lower horizontal line of contour C . We denote by $W_C[\delta y_1, \delta y_2]$ the Wilson loop with deformation δy_1 and δy_2 on the upper and lower line of C .

We will now compute the Wilson loop at strong coupling by using holography, that is we identify

$$\langle \text{tr} \rho^0 W_C \rangle = e^{i S_{\text{NG}}} \quad (3.19)$$

where ρ^0 is the density matrix [114], S_{NG} being the Nambu-Goto action, with the boundary of the string world sheet being the curve C . That is the string world sheet ends on the world line (3.13) of the heavy parton. If $X^\mu : (\sigma, \tau) \rightarrow (t, \zeta, x, y, z, \psi, \phi_1, \phi_2, \theta_1, \theta_2)$ is a mapping from string world sheet to ten-dimensional geometry given by (2.23) with $\zeta = 1/r$, then with parametrization $\sigma = \zeta, t = \tau$, we have

$$\begin{aligned} x(t, \zeta) &= vt + \bar{x}(\zeta) \\ z(t, \zeta) &= y(t, \zeta) = \delta y(t, \zeta) \end{aligned} \quad (3.20)$$

where \bar{x} is the unperturbed solution for the mapping. Now using background metric of the form (2.91) with $g_{rr} = H$, the Nambu-Goto action up to quadratic order in the perturbation δy gives

$$S_{\text{NG}} = \frac{1}{2\pi\alpha'} \int d\zeta dt \sqrt{\frac{H}{\zeta^4} + \frac{g}{h} [(\bar{x}')^2 + 2\delta y'^2] - \frac{H}{g\zeta^4} [v^2 + 2\delta \dot{y}^2]} \quad (3.21)$$

where we approximated $g_i = g$ and prime means derivative with respect to ζ while dot means derivative with respect to 't', $\dot{A} \equiv \frac{dA}{dt}$, $A' \equiv \frac{dA}{d\zeta}$. Minimizing this action with respect to \bar{x} with ignoring $\mathcal{O}(\delta y)$ gives the solution

$$\bar{x}'^2 = \frac{h^2 C^2 v^2 H (g - v^2)}{g^2 \zeta^4 (g - C^2 v^2 h)} \quad (3.22)$$

where C is a constant of integration.

Now to solve for the transverse fluctuation δy , note that the pullback metric $f_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu G_{\mu\nu}$ is off diagonal with components

$$f_{tt} = h(\zeta)^{-1/2} \left(-\frac{1}{\gamma^2} + \zeta^4 / \zeta_h^4 + \delta \dot{y}^2 \right)$$

$$\begin{aligned}
f_{t\zeta} &= \frac{1}{\sqrt{h}}(v\bar{x}') + \frac{1}{\sqrt{h}}\delta\dot{y}\delta y' \\
f_{\zeta\zeta} &= \frac{H\sqrt{h(\zeta)}}{\zeta^4 g(\zeta)} + \frac{1}{\sqrt{h}}(\bar{x}'^2 + \delta y'^2)
\end{aligned} \tag{3.23}$$

with $\delta\dot{y} \equiv d\delta y/dt$, $\delta y' \equiv d\delta y/d\zeta$, $g(\alpha) = 1 - \alpha^4/\zeta_h^4$ for any α and $\gamma = 1/\sqrt{1-v^2}$. This pullback metric can be diagonalized at zeroth order in y with the reparametrization

$$\begin{aligned}
\hat{t} &= \frac{1}{\sqrt{\gamma}}(t + F(\zeta)) \\
\hat{\zeta} &= \sqrt{\gamma}\zeta \\
\frac{dF}{d\zeta} &= -\frac{v\bar{x}'}{g(\zeta) - v^2}
\end{aligned} \tag{3.24}$$

The resulting pullback metric is

$$\begin{aligned}
f_{\hat{t}\hat{t}} &= \frac{1}{\sqrt{h(\zeta)}\gamma} \left(-g(\hat{\zeta}) + \delta\dot{\hat{y}}^2 \right) \\
f_{\hat{t}\hat{\zeta}} &= \mathcal{O}(\delta y^2); \\
f_{\hat{\zeta}\hat{\zeta}} &= \frac{1}{\gamma} \left(\frac{H(\zeta)\sqrt{h(\zeta)}}{\zeta^4(g(\zeta) - C^2 v^2 h(\zeta))} + \frac{1}{\sqrt{h(\zeta)}}\delta\dot{\hat{y}}'^2 \right. \\
&\quad \left. - \frac{\delta\dot{\hat{y}}^2 v^4 h(\zeta)^2 C^2 H(\zeta)}{g(\zeta)^2 \sqrt{h(\zeta)} \zeta^4 (g(\zeta) - v^2)(g(\zeta) - C^2 v^2 h(\zeta))} \right)
\end{aligned} \tag{3.25}$$

Here $\delta\hat{y} = \sqrt{\gamma}\delta y$, $\delta\dot{\hat{y}} = \frac{d\delta\hat{y}}{dt}$ and $\delta\hat{y}' = \frac{d\delta\hat{y}}{d\hat{\zeta}}$. Observe that the pullback metric has a horizon at $\hat{\zeta} = \zeta_h$ which means at $\zeta = \zeta_h/\sqrt{\gamma} < \zeta_h$ and thus the pullback has larger horizon radius than the space-time metric. Thus the world sheet only extends in the radial direction from $\zeta = 0$ to $\zeta = \zeta_h/\sqrt{\gamma}$. Also observe that at the horizon $g - C^2 v^2 h = 0$ by our choice of C and thus indeed $f_{\hat{\zeta}\hat{\zeta}} = \infty$ at the horizon. Now with this reparametrization, the terms quadratic order in $\delta\hat{y}$ in Nambu-Goto action become

$$\begin{aligned}
S_{NG,\delta y}^{[2]} &= \frac{1}{2\pi\alpha'} \int d\hat{\zeta} d\hat{t} \left[\frac{g(\hat{\zeta})\hat{\zeta}^2}{2\gamma^2 h(\zeta)\sqrt{1+\mathcal{A}}} \delta\hat{y}'^2 - \frac{1 + \gamma^2 h(\zeta)v^4 C^2}{2\hat{\zeta}^2 g(\hat{\zeta})} \delta\dot{\hat{y}}^2 \right] \\
\mathcal{A} &= \frac{g(\hat{\zeta})H(\zeta)}{g - C^2 v^2 h(\zeta)} - 1
\end{aligned} \tag{3.26}$$

If we denote $\delta\hat{y}(\hat{t}, \hat{\zeta}) = \int d\hat{\omega} e^{i\hat{\omega}\hat{t}} \delta\hat{y}(\hat{\omega})\hat{Y}(\hat{\zeta})$, then the equation of motion can be written as

$$\begin{aligned}\hat{Y}'' + \frac{B'}{B}\hat{Y}' - \frac{D}{B}\hat{Y} &= 0 \\ B &= \frac{g(\hat{\zeta})\hat{\zeta}^2}{\gamma^2 h(\zeta)\sqrt{1+\mathcal{A}}} \\ D &= -\frac{\hat{\omega}^2(1+\gamma^2 h(\zeta)v^4 C^2)}{2\hat{\zeta}^2 g(\hat{\zeta})}\end{aligned}\tag{3.27}$$

We try solution to (3.27) of the form

$$\begin{aligned}\hat{Y} &= g(\hat{\zeta})^\beta \mathcal{F} \\ \mathcal{F} &= (1 + \beta \mathcal{H})\end{aligned}\tag{3.28}$$

with β being a constant and we have written \mathcal{F} only up to linear order in β which is sufficient for what is to follow. Now from equation (3.27) near horizon $\hat{\zeta} \rightarrow \zeta_h$ and only considering up to quadratic order in β , we can isolate the most divergent terms to obtain

$$\begin{aligned}\beta^2 &= -\frac{\hat{\omega}^2 \bar{h} \sqrt{1+\tilde{\mathcal{A}}}(1+\zeta_h^4 \bar{h} v^4 C^2)}{g'(\zeta_h)} \\ \Rightarrow \beta &= \pm i \frac{\hat{\omega}}{4\pi T} (1 + \mathcal{E})\end{aligned}\tag{3.29}$$

where $\tilde{\mathcal{A}} = \mathcal{A}(\hat{\zeta} = \zeta_h)$, \mathcal{E} is of $\mathcal{O}(g_s M_{\text{eff}}^2/N, g_s^2(N_f^{\text{eff}}))^2$ and $\bar{h} = h(\zeta = \zeta_h/\sqrt{\gamma})$ while $T = \frac{\zeta_h}{\pi h(\zeta_h)}$ is the temperature associated with the space time black-hole (not the world sheet black hole). This determines our constant β and now we can solve (3.27) order by order in $\hat{\omega}$. For the purpose of calculating κ_T , we will eventually take the zero frequency $\omega \rightarrow 0$ limit (albeit after dividing by ω), thus it is sufficient to solve (3.27) only up to linear order in ω . With the form of the solution as in (3.28), the equation of motion (3.27) gives the following equation for \mathcal{H} ,

$$\mathcal{H}'' + \frac{B'}{B}\mathcal{H}' + \frac{g(\hat{\zeta})''}{g(\hat{\zeta})} - \frac{\tilde{h}'g(\hat{\zeta})'}{\tilde{h}g(\hat{\zeta})} = 0\tag{3.30}$$

where $\tilde{h} = 2\gamma^2 h(\zeta) \frac{\sqrt{1+\mathcal{A}}}{\zeta^2}$ and prime denotes a derivative with respect to $\hat{\zeta}$. From the above form of the equation, we observe with $W = \mathcal{H}'$ we can write it as

$$\begin{aligned} [WB]' &= -B \left(\frac{g(\hat{\zeta})''}{g(\hat{\zeta})} - \frac{\tilde{h}'g(\hat{\zeta})'}{\tilde{h}g(\hat{\zeta})} \right) \\ \Rightarrow \mathcal{H}' &= -\frac{1}{B} \int d\hat{\zeta} \ B \left(\frac{g(\hat{\zeta})''}{g(\hat{\zeta})} - \frac{\tilde{h}'g(\hat{\zeta})'}{\tilde{h}g(\hat{\zeta})} \right) \end{aligned} \quad (3.31)$$

We will impose the boundary condition $Y(0) = 1$ which implies $\mathcal{H}(0) = 0$. Now using the solution for \hat{Y} and taking appropriate linear combinations to build the type ‘1’ and ‘2’ fields $\delta y_1, \delta y_2$ as in [114], we can write the boundary action after integrating the Nambu-Goto action and the result is equation (3.51) of [114] but \hat{Y} is replaced with our solution, $\hat{\omega}, \omega$ replaced by $\hat{\omega}/\pi T, \omega/\pi T$ and $R = 1$. Finally from the boundary action we can obtain the Greens function G_{ij} and the result for the diffusion coefficient is

$$\kappa_T = \sqrt{\gamma g_s \bar{N}_{\text{eff}}} \pi T^3 (1 + \mathcal{B}) \quad (3.32)$$

where \mathcal{B} is of $\mathcal{O}(g_s M_{\text{eff}}^2/N, g_s^2(N_f^{\text{eff}}))^2$ and higher, while \bar{N}_{eff} is the number of effective degrees of freedom for the boundary gauge theory

$$\begin{aligned} \bar{N}_{\text{eff}} &= N \left(1 + \frac{27g_s^2 M_{\text{eff}}^2 N_f^{\text{eff}}}{32\pi^2 N} - \frac{3g_s M_{\text{eff}}^2}{4\pi N} + \left[\frac{3g_s M_{\text{eff}}^2}{4\pi N} - \frac{9g_s^2 M_{\text{eff}}^2 N_f^{\text{eff}}}{16\pi^2 N} \right] \log r_0 \right. \\ &\quad \left. + \frac{9g_s^2 M_{\text{eff}}^2 N_f^{\text{eff}}}{8\pi^2 N} \log^2 r_0 \right) \end{aligned} \quad (3.33)$$

where r_0 is as in section 2.4 i.e. it is the scale where warp factor changes to inverse power series from logarithm. Note for duality to hold, N must be quite large, making N_{eff} quite large. For $N_f^{\text{eff}} = M_{\text{eff}} = 0$, we get back the value of κ_T as computed in [114]. However for a non conformal field theory with fundamental matter - which is more relevant for QCD - $M_{\text{eff}} \neq 0, N_f^{\text{eff}} \neq 0$, and our analysis thus generates a correction to the AdS/CFT result.

3.1.3 Wake

In the previous sections we computed the drag force on the quark along with the broadening of its momentum. Clearly a moving quark should leave some disturbance

in the surrounding media. This disturbance is called the *wake* of the quark. In order to quantify the wake left behind by a fast moving quark in the Quark Gluon Plasma, we need to compute the stress tensor T^{pq} , $p, q = 0, 1, 2, 3$ of the entire system. We expect a cone like disturbance in the medium with the quark located at the tip of the cone and this cone becomes apparent when one plots the stress tensor as a function of location of the fast quark. We would like to compute the energy momentum tensor the medium including the fast quark, then subtract the contribution from the quark to obtain the disturbance left behind in the medium. Our goal therefore would be to compute:

$$T_{\text{medium+quark}}^{pq} - T_{\text{quark}}^{pq} \quad (3.34)$$

where the first term is basically the energy-momentum tensor obtained from dual geometry considering back reaction of a moving string i.e $T_{\text{background+string}}^{pq}$. Similarly the second term is the energy momentum tensor of the string i.e T_{string}^{pq} restricted to four-dimensional space-time. This is similar to the analysis done in [69] for the AdS case. For our case the above idea, although very simple to state, will be rather technical because of the underlying RG flow in the dual gauge theory side. Our second goal would then be to see how much we differ from the AdS results once we go from CFT to theories with running coupling constants.

For a strongly coupled QGP, we will apply the gauge/gravity duality to compute T^{pq} of QGP using the supergravity action. In the ten dimensional bulk geometry, we introduce an additional string moving with some velocity and this string is dual to the fast parton creating the wake. The total metric takes the following form:

$$\begin{aligned} G_{ij} &= g_{ij} + \kappa l_{ij} \\ l_{ij} &\equiv l_{ij}(r, x, y, z, t) \end{aligned} \quad (3.35)$$

where g_{ij} is the background metric and l_{ij} ($i, j = 0, \dots, 9$) denote the perturbation from the moving string source (with $\kappa \rightarrow 0$). In order to compute T^{pq} , we need to write the supergravity action as a functional of the perturbation l_{pq} . We have $\mathcal{O} = T^{pq}$ in

(2.51) and thus $\phi_0 = \kappa l_{pq}$ is the source in the partition function (2.49). It follows that

$$\langle T^{pq} \rangle = \frac{1}{\kappa} \frac{\delta S}{\delta l_{pq}} \Big|_{\kappa l_{pq}=0} \quad (3.36)$$

where $S \equiv S_{\text{total}} + S_{\text{GH}} + S_{\text{counterterm}}$ as discussed in section **2.3.1** where now S_{total} includes the DBI action for the string.

Minimizing S_{total} , we obtain the equation of motion for G_{ij} :

$$R_{ij} (g_{\alpha\beta} + \kappa l_{\alpha\beta}) - \frac{1}{2} (g_{ij} + \kappa l_{ij}) R (g_{\alpha\beta} + \kappa l_{\alpha\beta}) = T_{ij}^{\text{string}} + T_{ij}^{\text{fluxes}} + T_{ij}^{(p,q)7} \quad (3.37)$$

where the T_{ij}^{fluxes} come from the five-form fluxes $F_{(5)}$ (that give rise to the AdS_5 part) and the remnant of the H_{NS}, H_{RR} and the axio-dilaton along the radial r direction (that give rise to the deformation of the AdS_5 part). . The effect of $T_{ij}^{(p,q)7}$ will not be substantial if we take it as a probe in this background whereas the strings stress tensor is given explicitly by

$$\begin{aligned} T_{\text{string}}^{ij}(x) &= \frac{\delta S_{\text{string}}}{\delta G_{ij}} \\ &= \int d\sigma d\tau \left(\frac{2\dot{X} \cdot X' \dot{X}^i X'^j - X'^i X'^j \dot{X}^2 - \dot{X}^i \dot{X}^j X'^2}{2\sqrt{-\det f}} \right) \delta^{10}(X - x) \end{aligned} \quad (3.38)$$

where X is the mapping from string world sheet to space time, and we can consider the string profile given by (3.2).

The Einstein equations can be worked out if one considers the effects of all the background fluxes in our theory. The result of such an analysis can be presented in powers of κ . For our case we are only interested in back reactions that are linear in κ . To this order the equation of motion satisfied by $l_{\alpha\beta}$ is determined by expanding (3.37) in the following way:

$$\kappa \left(\Delta_{ij}^{\alpha\beta} - \mathcal{B}_{ij}^{\alpha\beta} - \mathcal{A}_{ij}^{\alpha\beta} \right) l_{\alpha\beta} = T_{ij}^{\text{string}} \quad (3.39)$$

where $\Delta_{ij}^{\alpha\beta}, \alpha, \beta = 0, \dots, 9$ is an operator whereas $\mathcal{B}_{ij}^{\alpha\beta}$ and $\mathcal{A}_{ij}^{\alpha\beta}$ are functions of r , the radial coordinate⁵. We have been able to determine the form for the operator

⁵There will be another contribution from the (p, q) seven branes in the background, although for small $g_s N_f$ these are sub leading.

$\Delta_{ij}^{\alpha\beta}$ for any generic perturbation $l_{\alpha\beta}$ in five dimensions that is by setting $l_{ij} = 0$ for $i, j = 5, \dots, 9$. The resulting equations are rather long and involved; and we give them in the Appendix B of [57]. For the functions $\mathcal{A}_{ij}^{\alpha\beta}$ and $\mathcal{B}_{ij}^{\alpha\beta}$ we have worked out a toy example in Appendix C of [57] with only diagonal perturbations. For off diagonal perturbations we need to take an inverse of a 5×5 matrix to determine the functional form. We shall provide details of this in the following. The variables defined in (3.39) are given as:

$$\begin{aligned}
\Delta_{ij}^{\alpha\beta} &= \left(\frac{\delta R_{ij}}{\delta g_{\alpha\beta}} \right) - \frac{1}{2} g_{ij} \left(\frac{\delta R}{\delta g_{\alpha\beta}} \right) - \frac{1}{2} R \delta_{\mu\alpha} \delta_{\nu\beta} \\
\mathcal{A}_{ij}^{\alpha\beta} &= 5 \sum_{b,c,d,\dots} F_{(5)\mu bcda} F_{(5)\nu b'c'd'a'} g^{bb'} g^{cc'} g^{dd'} g^{aa'} g^{a'\beta} \\
\mathcal{B}_{ij}^{\alpha\beta} &= -\frac{5}{8} \sum_{a,b,c,d,\dots} F_{(5) nabcd} F_{(5)n'a'b'c'd'} g^{aa'} g^{bb'} g^{cc'} g^{dd'} g^{nn'} (g_{ij} g^{\alpha\beta} - \delta_{\mu}^{\alpha} \delta_{\nu}^{\beta}) \\
&\quad - \frac{1}{4} \sum_{i=1}^4 g_{ij} F_a^{(i)} F_b^{(i)} g^{aa} g^{bb} + \sum_{i=1}^4 F_r^{(i)} F_r^{(i)} g^{rr} \delta_{\mu}^{\alpha} \delta_{\nu}^{\beta}
\end{aligned} \tag{3.40}$$

where we have given the most generic form in (3.40) above for a five dimensional perturbation. Furthermore, $\frac{\delta R_{ij}}{\delta g_{\alpha\beta}}$ and $\frac{\delta R}{\delta g_{\alpha\beta}}$ are operators and not functions.

Once (3.39) is solved, with the exact solutions l_{ij} , we can write the action as a functional of l_{ij} . We can then integrate the radial direction along with the internal directions and add appropriate Gibbons-Hawking terms along with counterterms to renormalize the action if necessary. The renormalization and calculation of the stress tensor was described in some detail in section 2.3.1 and 2.3.2. The final result for the stress tensor T^{pq} with $\kappa l_{ij} = \phi_m$ is given by (2.90). Knowing all the solutions l_{ij} , one can easily evaluate the stress tensor and finally plot the wake. Here we have outlined the procedure and an exact calculation will be done in our future work.

3.2 Transport Coefficients

At low energies, an effective theory of fluids is hydrodynamics which describes the kinematics of the fluid and uniquely determines its stress energy tensor. As dissipation is allowed, the theory is not described in terms of action but in terms of equation of motion, namely the equation which guarantees conservation of energy-momentum

tensor

$$\nabla_\mu T^{\mu\nu} = 0 \quad (3.41)$$

where $\mu, \nu = 0, 1, 2, 3$. There are four equations above and they can be solved by four independent variables [116][117][118]. If we assume local thermal equilibrium then the state of the system can be uniquely determined by the local fluid velocity $u^\mu(x^\mu)$ and local temperature $T(x^\mu)$. There are actually three independent components of the four velocity u^μ as $u^\mu u_\mu = -1$ fixes one of the components and the fourth independent variable is the temperature. Using these four independent variables, we can write the expression for stress tensor that satisfies equation (3.41) and at zeroth order in derivatives of u^μ we obtain the stress tensor of ideal fluid:

$$T_{\mu\nu}^{[0]} = (\epsilon + P)u_\mu u_\nu + P g_{\mu\nu} \quad (3.42)$$

where ϵ is the energy, P is the pressure and $g_{\mu\nu}$ is the metric of four dimensional space time where the fluid lies.

Now at linear order in derivatives, the allowed terms are restricted by rotational symmetry and only nonzero components are $T_{ij}^{[1]}, i, j = 1, 2, 3$

$$T_{ij}^{[1]} = P_{i\alpha} P_{j\beta} \left[\eta \left(\nabla^\alpha u^\beta + \nabla^\beta u^\alpha \right) + \left(\zeta - \frac{2}{3}\eta \right) g^{\alpha\beta} \nabla \cdot u \right] \quad (3.43)$$

where $P_{i\alpha} = g_{i\alpha} - u_i u_\alpha$ and the coefficients of the two terms determine the transport coefficients shear viscosity η and bulk viscosity ζ [117]. Thus (3.43) is the defining equation for viscosity.

Physically shear viscosity measures the mixing between two layers of a fluid [57]. More viscous the fluid, the faster momentum can be transferred from a layer to the next. Somewhat counter intuitive is the fact that the stronger the coupling in the fluid, the less the shear viscosity. This is because the rate of mixing is controlled by the mean free path. When the mean free path is small compared to the flow velocity variation of the two laminas, the layers cannot easily mix since the exchange of particles is limited to the small volume near the interface of the two laminas: Most particles in the fluid just flows along as if there is no other layers nearby. On the

other hand, if the mean free path is comparable to the typical size of the flow velocity variation, then mixing between different layers can proceed relatively quickly. For an ideal fluid, laminas of fluid do not interact at all and we have zero viscosity.

Transport coefficient such as shear viscosity becomes crucial in understanding physics of quark gluon plasma. In particular, as already mentioned in the introduction, in a heavy ion collision the overlapping region of the two nuclei is of elliptical shape with different short and long axes. Thus the plasma formed undergoes elliptic flow due to difference in pressure along the long and short axes. On the other hand, a large viscosity would mean greater interaction between the layers of fluid which would quickly equilibrate the system and one would observe very little elliptic flow. But the data from RHIC is well described by ideal hydrodynamics with zero shear viscosity and shows strong elliptic flow [19]-[22]. This means the fluid created in heavy ion collisions has small viscosity and thus is strongly coupled. Thus one cannot apply conventional perturbative field theory techniques to compute the transport coefficients of the plasma.

However in the regime of strong coupling, certain gauge theories can be described by dual supergravity. In particular for the gauge theories described in section **2.2-2.4**, which have several features common to QCD, one can compute the transport coefficients at strong coupling using dual geometry. The result we obtain can then be suggestive of the viscosities of QGP. In the following sections we compute shear viscosity η for gauge theories using dual supergravity and the ratio η/s which appears in various experimental observations.

3.2.1 Shear Viscosity

In this section we will present our calculation [57] of shear viscosity of the four dimensional theory following some of the recent works [146][120]. The shear viscosity described earlier can be obtained from correlation functions using the Kubo formula [116]:

$$\eta = \lim_{\omega \rightarrow 0} \frac{1}{2\omega} \int dt d^3x e^{i\omega t} \langle [T_{23}(x), T_{23}(0)] \rangle = - \lim_{\omega \rightarrow 0} \frac{\text{Im } G^R(\omega, 0)}{\omega} \quad (3.44)$$

where $G^R(\omega, \vec{q})$ is the momentum space retarded propagator for the operator T_{23} at finite temperature, defined by

$$G^R(\omega, \vec{q}) = -i \int dt d^3x e^{i(\omega t - \vec{x} \cdot \vec{q})} \theta(t) \langle [T_{23}(x), T_{23}(0)] \rangle \quad (3.45)$$

In the following, we will compute the Minkowski propagator following the conjecture made in [121][122] using dual gravity. Note that our prescription (2.49) computes a path integral with a classical action S_{SUGRA} , unaware of the ordering of the operators whose expectation value is being computed. Therefore computing any commutator is subtle here. Hence, we compute only the correlator $\langle T_{23}(\tau, \vec{x}) T_{23}(0, \vec{x}) \rangle$ and using the conjecture in [121], relate this to the retarded Greens function. However before we compute this explicitly, let us evaluate the higher order corrections to the effective action from the wrapped D7 brane in our theory.

In the case of a single D7 brane, the disc level action contains the term [123][124]:

$$S_{\text{D7}}^{\text{disc}} = \frac{1}{192\pi g_s} \cdot \frac{1}{(4\pi\alpha')^2} \int_{M^8} [C_4 \wedge \text{tr} (R \wedge R) - e^{-\phi} \text{tr} (R \wedge *R)] \quad (3.46)$$

where C_4 is the four-form, R is the curvature two-form, ϕ is the dilaton and M^8 is a non-trivial eight manifold which is the world-volume of the D7 brane. The action is $SL(2, \mathbf{Z})$ invariant which was shown by doing an explicit analysis [123][124]. Since for our case the D7 wrap a non-trivial four-cycle, we can dimensionally reduce it over the four-cycle and obtain the following action:

$$S_{\text{D7}}^{\text{disc}} = \frac{1}{16\pi^2} \int_{M^4} \text{Re} [\log \eta(\tau) \text{tr} (R \wedge *R - iR \wedge R)] \quad (3.47)$$

where $\eta(\tau)$ is the Dedekind function, and τ is the modular parameter defined as follows:

$$\tau = \frac{1}{g_s(4\pi\alpha')^2} \left(\int_{S^4} C_4 + i\mathcal{V}_4 \right) \equiv \frac{1}{g_s} (\tau_1 + i\tau_2) \quad (3.48)$$

with \mathcal{V}_4 being the volume of the four-cycle on which we have the wrapped D7 brane. In fact the above action can be *derived* from the following action that has two parts, CP-even and CP-odd [125]:

$$\frac{1}{32\pi^2} \int_{M^4} \log |\eta(\tau)|^2 \text{tr} (R \wedge *R) - \frac{i}{32\pi^2} \int_{M^4} \log \frac{\eta(\tau)}{\eta(\bar{\tau})} \text{tr} (R \wedge R) \quad (3.49)$$

where the first part is CP-even and the second part is CP-odd. To compare (3.49) with (3.47) note that the Dedekind η function has the following expansion with $q \equiv e^{2\pi i \tau}$:

$$\begin{aligned} \log|\eta(\tau)|^2 &= -\frac{\pi}{6}\tau_2 - \left[q + \frac{3q^2}{2} + \frac{4q^3}{3} + \dots + \text{c.c} \right] \\ \log \frac{\eta(\tau)}{\eta(\bar{\tau})} &= +\frac{i\pi}{6}\tau_1 - \left[q + \frac{3q^2}{2} + \frac{4q^3}{3} + \dots - \text{c.c} \right] \end{aligned} \quad (3.50)$$

Combining everything we see that, up to powers of q (3.49) and (3.47) are equivalent. However writing the action in terms of (3.49) instead of (3.47) has the following advantage: from D7 point of view (3.49) captures the D3 instanton corrections in the system [125]-[129]. But a deeper reason for writing the action as (3.49) is that the CP-even and CP-odd terms can be expanded further to account Gauss-Bonnet type interactions [125]:

$$S_{\text{CP-even}} = -\alpha_1 \int_{M^8} e^{-\phi} \mathcal{L}_{\text{GB}} - T_7 \int_{M^8} e^{-\phi} \left[\sqrt{G} - \frac{(4\pi^2 \alpha')^2}{24} \mathcal{L}_R + \mathcal{O}(\alpha'^4) \right] \quad (3.51)$$

where α_1 is a constant, and \mathcal{L}_{GB} and \mathcal{L}_R are respectively the Gauss-Bonnet and the curvature terms defined in the following way:

$$\begin{aligned} \mathcal{L}_{\text{GB}} &= \frac{\sqrt{G}}{32\pi^2} (R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} - 4R_{\alpha\beta} R^{\alpha\beta} + R^2) \\ \mathcal{L}_R &= \frac{\sqrt{G}}{32\pi^2} (R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} - 2R_{\alpha\beta} R^{\alpha\beta} - R_{ab\gamma\delta} R^{ab\gamma\delta} + 2R_{ab} R^{ab}) \end{aligned} \quad (3.52)$$

In the above note that the three curvature terms $R_{\alpha\beta} R^{\alpha\beta}$, $R_{ab} R^{ab}$ and R^2 are *not* the pull-backs of the bulk Ricci tensor. We have also used the notations (α, β) to denote the world-volume coordinates, and (a, b) to denote the normal bundle.

From the CP-even terms, the coefficient of $R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}$ is given by [57]:

$$c_3 \equiv \frac{e^{-\phi} \sqrt{G}}{32\pi^2} \left(\frac{4\pi^4 \alpha'^2}{3} - \alpha_1 \right) \quad (3.53)$$

which has an overall plus sign because α_1 in many cases is zero (see [125] for a discussion on this). However in general for certain exotic compactifications we can have $\alpha_1 < \frac{4\pi^4 \alpha'^2}{3}$. If we now compare this to [146] we see that c_3 , which is the coefficient of $R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}$ in [146], is indeed positive. This would clearly mean that

adding fundamental flavors lowers the viscosity to entropy bound!⁶

The CP-odd term on the other hand has a standard expansion [131]-[134],[123]-[125]:

$$S_{\text{CP-odd}} = T_7 \int_{M^8} \left(C_8 + \frac{\pi^2 \alpha'^2}{24} C_4 \wedge \text{tr } R \wedge R \right) \quad (3.54)$$

where the first term gives the dual axionic charge of the D7 brane. Combining (3.51) and (3.54) we get the full back reactions of the D7 brane up to $\mathcal{O}(\alpha'^2)$.

Having computed the back reactions of the embedded D7 brane, we can use this result to compute the shear viscosity. To start the analysis, we need the correlation function of $T_{23}(x)$ and $T_{23}(0)$ to use it in the Kubo formula (3.44). From gauge/gravity duality we know that switching on T_{23} in the gauge theory is equivalent to considering graviton modes along $x^2 = x$ and $x^3 = y$ directions. In the ten dimensional dual geometry with the metric of the form (2.91), we introduce graviton perturbations in the xy direction. We start with the ten dimensional SUGRA action with the perturbed metric and then integrate over the five internal compact directions to obtain an effective five dimensional action. The resulting five dimensional metric $ds_5^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu$ which minimizes the five dimensional action takes the following form:

$$\begin{pmatrix} \tilde{g}_{00} & \tilde{g}_{0x} & \tilde{g}_{0y} & \tilde{g}_{0z} & \tilde{g}_{0r} \\ \tilde{g}_{x0} & \tilde{g}_{xx} & \tilde{g}_{xy} & \tilde{g}_{xz} & \tilde{g}_{xr} \\ \tilde{g}_{y0} & \tilde{g}_{yx} & \tilde{g}_{yy} & \tilde{g}_{yz} & \tilde{g}_{yr} \\ \tilde{g}_{z0} & \tilde{g}_{zx} & \tilde{g}_{zy} & \tilde{g}_{zz} & \tilde{g}_{zr} \\ \tilde{g}_{r0} & \tilde{g}_{rx} & \tilde{g}_{ry} & \tilde{g}_{rz} & \tilde{g}_{rr} \end{pmatrix} = \frac{1}{\sqrt{\bar{h}(r)}} \begin{pmatrix} -g(r) & 0 & 0 & 0 & 0 \\ 0 & 1 & \phi(r, t) & 0 & 0 \\ 0 & \phi(r, t) & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{\bar{h}(r)}{g(r)} \end{pmatrix}$$

where we have ignored $\mathcal{O}(g_s^k N_f^k), k \geq 2$ terms (which is equivalent to setting $g_{rr} = 1$ in (2.91) and thus in five dimensions the effective g_{rr} is 1) and $\bar{h}(r)$ is now only a function of the radial coordinate. Note that \bar{h} is obtained from h appearing in (2.91) by integrating over the internal coordinates on which h depends. When h is

⁶The analysis here was motivated by discussions with Aninda Sinha. His paper [130] dealing with the violation of viscosity to entropy bound appeared recently and has some overlap with our analysis.

independent of any of the angles of internal space, like the case in AdS space, then $\bar{h} = h$. We have also taken the approximation $g_1 = g_2 = g(r)$ for simplicity and this approximation becomes exact if we ignore $\mathcal{O}(g_s N_f, g_s M^2/N)$ corrections to the black hole factors g_i .

Since our goal is to compute the Fourier transform of $\langle T_{23}(t, \vec{x}) T_{23}(0, \vec{x}) \rangle$, we can do this by first writing the supergravity action in momentum space, treating it as a functional of Fourier modes for $\phi(r, t)$ where:

$$\begin{aligned}\phi(r, t) &= \tilde{\phi}(r, t) \bar{\phi}(t) \equiv \int d\omega e^{-i\omega\tau} \phi(r, \omega) = \int d\omega e^{-i\omega\sqrt{g}t} \phi(r, \omega) \\ \phi(r, \omega) &= \tilde{\phi}(r, |\omega|) \bar{\phi}(\omega)\end{aligned}\tag{3.55}$$

where as before, we defined the Fourier transform using the curved space time $\tau \equiv \sqrt{g(r_c)} t$ and not simply t . Although this definition is precise for the theory at the cut-off $r = r_c$ only, we will use it also for any r because in the end we will only provide description at the boundary (i.e $r \rightarrow \infty$) where the results would be independent of the choice of the cut-off.

The way we proceed now is the following⁷. We consider the metric fluctuation as in (3.2.1) and plug this in the five dimensional effective action. Finally, we will call this resulting action as $S_{\text{SG}}^{(2)}$ where the subscript (2) involves writing the action in terms of quadratic $\phi(r, \omega)$. We do this as there exists a very useful relation for computing the shear viscosity (see for example [121][122]):

$$\lim_{\omega \rightarrow 0} \text{Im } G_{11}^{\text{SK}}(\omega, \vec{0}) = \lim_{\omega \rightarrow 0} \frac{2T}{\omega} \text{Im } G^R(\omega, \vec{0})\tag{3.56}$$

where G_{ij}^{SK} is the Schwinger-Keldysh propagator [121]-[135]. Comparing this with our earlier Kubo formula (3.44), we get the following expression for shear viscosity:

$$\eta = -\frac{1}{2T} \lim_{\omega \rightarrow 0} \text{Im } G_{11}^{\text{SK}}(\omega, \vec{0})\tag{3.57}$$

Thus if we can write our effective supergravity action in the following way:

$$S_{\text{SG}}^{(2)}[\phi(r_b, \omega)] = \frac{1}{2} \int \frac{d\omega d^3q}{(2\pi)^4} \phi_i(r_b, \omega) G_{ij}^{\text{SK}}(|\omega|, \vec{q}) \phi_j(r_b, -\omega)\tag{3.58}$$

⁷This is similar to the procedure of [146]. Notice however that the theory considered by [146] has no running but contains higher curvature-squared corrections.

where r_b is a specified point on the boundary. Then taking the $G_{11}^{\text{SK}}(\omega, \vec{0})$ part and using (3.57) we can easily obtain the shear viscosity⁸. In other words, we will be taking two functional derivatives of $S_{\text{SG}}^{(2)}[\phi(r_b, \omega)]$ with respect to $\phi(r_b, \omega)$ and thus are interested about terms quadratic in $\phi(r_b, \omega)$ in the action. Of course, in real time formalism, we are concerned with the Schwinger-Keldysh propagator G_{ij}^{SK} of the doublet fields $\phi_i(r, t), \phi_j(r, t)$. In the context of gauge/gravity duality, we follow the procedure outlined by [135] for AdS/CFT correspondence and treat $\phi_1(r, t), \phi_2(r, t)$ as the perturbation $\phi(r, t)$ and its doublet in the four dimensional Minkowski space⁹. In ten dimensional gravity theory, $\phi_1(r) = \phi(r)$ is the field in the R quadrant of the Penrose diagram and $\phi_2(r)$ is the field in the L quadrant. For more details see [135]-[140].

To be more precise, our aim is to get the effective action in the form (3.58). To this effect we take our metric (3.2.1) and plug it in the five dimensional effective action with net result:

$$S_{\text{SG}}^{(2)} = \frac{1}{8\pi G_N \sqrt{g(r_c)}} \int \frac{d\omega d^3q}{(2\pi)^4} \int_{r_h}^{r_c} dr \left[A(r) \phi(r, -\omega) \phi''(r, \omega) + B(r) \phi'(r, -\omega) \phi'(r, \omega) \right. \\ \left. + C(r) \phi(r, -\omega) \phi'(r, \omega) + D(r) \phi(r, -\omega) \phi(r, \omega) \right] \quad (3.59)$$

where prime denotes derivative with respect to r and the explicit expressions for A, B, C, D are given in Appendix E of [57]. The five dimensional Newton's constant is given by:

$$G_N \equiv \frac{\kappa_{10}^2 L^5}{4\pi V_{T^{1,1}}} \quad (3.60)$$

where volume of $T^{1,1}$ i.e $V_{T^{1,1}}$ is dimensionful and κ_{10} is proportional to ten dimensional Newton's constant.

⁸Notice that there would be an overall volume factor of $T^{1,1}$ that would appear with the effective action. This factor just modifies the Newton's constant in five dimensions and does not effect dimensionless ratios like η/s , as we shall find out.

⁹ Although our background is a deformation of the AdS space, the arguments of [135] also apply here. We can still consider the ϕ_1 perturbations to compute the Schwinger-Keldysh propagator because by definition a propagator is what appears sandwiched between the fields. See also [140] for a generic approach.

The fluctuation $\phi(r, \omega)$ satisfies the following Euler-Lagrange equation of motion:

$$\phi''(r, \omega) + \frac{A'(r) - B'(r)}{A(r) - B(r)} \phi'(r, \omega) + \frac{2D(r) - C'(r) + A''(r)}{2[A(r) - B(r)]} \phi(r, \omega) = 0 \quad (3.61)$$

which we can derive from 3.59 by minimizing it. Once we plug in the values of A, B etc., the above Euler-Lagrange equation takes the following form:

$$\begin{aligned} \phi''(r, \omega) + \left[\frac{g'(r)}{g(r)} + \frac{5}{r} + \mathcal{M}(r) \right] \phi'(r, \omega) + \left[\frac{\omega^2 g(r_c) \bar{h}(r)}{g(r)^2} + \mathcal{J}(r) \right] \phi(r, \omega) &= 0 \\ \bar{h}(r) \equiv \frac{L^4}{r^4} \left\{ 1 + \frac{3g_s N_f^2}{2\pi N} \left[1 + \frac{3g_s N_f}{2\pi} \left(\log r + \frac{1}{2} \right) - \frac{g_s N_f}{4\pi} \right] \log r \right\} \end{aligned} \quad (3.62)$$

where $\mathcal{J}(r)$ and $\mathcal{M}(r)$ appear due to seven branes and fluxes in the geometry¹⁰. As before, primes in (3.61) and (3.62) denote derivatives with respect to the five dimensional radial coordinate r .

Now as we mentioned above in (3.55), $\phi(r, \omega)$ can be decomposed in terms of $\tilde{\phi}(r, |\omega|)$ and $\bar{\phi}(\omega)$. Then as a trial solution, just like in [146], we first try $\tilde{\phi}(r, \omega) = g(r)^\gamma$ and look at (3.62) for r near the horizon r_h where $g(r) \rightarrow 0$. Plugging this in (3.62) with $g(r) = 0$ we obtain :

$$\begin{aligned} \gamma &= \pm i |\omega| \sqrt{\frac{\bar{h}(r_h) g(r_c)}{16}} r_h \\ &= \pm i \frac{|\omega|}{4\pi T_c} \end{aligned} \quad (3.63)$$

where in the last step we have used the definition of temperature T_c as in (2.45).

To get the solution with for general r , where $g(r) \neq 0$, we propose the following ansatz for the solution to (3.62):

$$\phi(r, \omega) = g(r)^{\pm i \frac{|\omega|}{4\pi T_c}} F(r, |\omega|) \bar{\phi}(\omega) \quad (3.64)$$

Plugging this in (3.62) we see that the equation satisfied by $F(r, |\omega|)$ can be expressed in terms of γ and γ^2 in the following way:

$$F''(r, |\omega|) + \left(\frac{g'(r)}{g(r)} + \frac{5}{r} + \mathcal{M}(r) \right) F'(r, |\omega|) + \left(\frac{|\omega|^2 g(r_c) \bar{h}}{g^2(r)} + \mathcal{J}(r) \right) F(r, |\omega|) = 0 \quad (3.65)$$

¹⁰In special cases we expect $\mathcal{J}(r)$ and \mathcal{M} to vanish (see for example [141]). However when this is not the case, as possible for non-trivial UV completions of our model, we could expect a non-minimally coupled scalar field.

$$+ \gamma \left\{ \frac{2g'(r)}{g(r)} F'(r, |\omega|) + \left[\frac{g''(r)}{g(r)} + \left(\frac{5}{r} + \mathcal{M} \right) \frac{g'(r)}{g(r)} \right] F(r, |\omega|) \right\} + \gamma^2 \frac{g'^2(r)}{g^2(r)} F(r, |\omega|) = 0$$

where the γ^2 terms come from both the last term in the above equation as well as the $|\omega|^2$ term above. Furthermore, note that the source $\mathcal{J}(r) \sim \mathcal{O}(g_s) + \mathcal{O}(g_s^2)$, so in the limit $g_s \rightarrow 0$ we find that (3.65) has a solution of the form $F(r, |\omega|) = c_1 + c_2 g(r)^{-2\gamma}$ with c_1, c_2 constants. Then we expect the complete solution for $g_s \neq 0$ to be $F(r, |\omega|) = c_1 + c_2 g(r)^{-2\gamma} + f(r, |\omega|)$. Demanding that $F(r, |\omega|)$ be regular at the horizon $r = r_h$ forces $c_2 = 0$ as $g(r_h) = 0$. We choose $c_1 = 1$ and $f = \mathcal{G} + \gamma \mathcal{H} + \gamma^2 \mathcal{K} + \dots$ as a series solution in γ . Then our ansatz for the solution to (3.65) becomes

$$F(r, |\omega|) = 1 + \mathcal{G}(r) + \gamma \mathcal{H}(r) + \gamma^2 \mathcal{K}(r) + \dots \quad (3.66)$$

Once we plug in the ansatz (3.66) in (3.65) we see that the resulting equation can be expressed as a series in γ :

$$\begin{aligned} & \mathcal{G}'' + \left(\frac{g'}{g} + \frac{5}{r} + \mathcal{M} \right) \mathcal{G}' + \mathcal{J}(1 + \mathcal{G}) \\ & + \gamma \left\{ \mathcal{H}'' + \left(\frac{g'}{g} + \frac{5}{r} + \mathcal{M} \right) \mathcal{H}' + \mathcal{J}\mathcal{H} + \frac{2g'}{g} \mathcal{G}' + \left[\frac{g''}{g} + \left(\frac{5}{r} + \mathcal{M} \right) \frac{g'}{g} \right] (1 + \mathcal{G}) \right\} \\ & + \gamma^2 \left\{ \mathcal{K}'' + \left(\frac{g'}{g} + \frac{5}{r} + \mathcal{M} \right) \mathcal{K}' + \mathcal{J}\mathcal{K} + \frac{2g'}{g} \mathcal{H}' + \left[\frac{g''}{g} + \left(\frac{5}{r} + \mathcal{M} \right) \frac{g'}{g} \right] \mathcal{H} \right. \\ & \quad \left. + \left(\kappa_0 + \frac{g'^2}{g^2} \right) (1 + \mathcal{G}) \right\} \\ & + \gamma^3 \left\{ \frac{2g'}{g} \mathcal{K}' + \left[\frac{g''}{g} + \left(\frac{5}{r} + \mathcal{M} \right) \frac{g'}{g} \right] \mathcal{K} + \left(\kappa_0 + \frac{g'^2}{g^2} \right) \mathcal{H} + \dots \right\} + \mathcal{O}(\gamma^4) = 0 \end{aligned} \quad (3.67)$$

where we have avoided showing the explicit r dependences of the various parameters to avoid clutter. We have also defined κ_0 in terms of the variables of (3.63) in the following way:

$$\kappa_0 \equiv -\frac{16}{\mathcal{T}^2 g^2} \quad (3.68)$$

Although the above equation (3.67) may look formidable there is one immediate simplification that could be imposed, namely, putting the coefficients of $\gamma^0, \gamma, \gamma^2, \dots$ individually to zero. This is possible because one can view γ to be an arbitrary

parameter that can be tuned by choosing the graviton energy ω or the temperature T_c . This means that the zeroth order in γ we will have the following equation:

$$\mathcal{G}''(r) + \left[\frac{g'(r)}{g(r)} + \frac{5}{r} + \mathcal{M}(r) \right] \mathcal{G}'(r) + \mathcal{J}(r)[1 + \mathcal{G}(r)] = 0 \quad (3.69)$$

In the above equation observe that the source $\mathcal{J}(r)$, for cases where it is non-zero, has a complicated structure with logarithms and powers of r . To simplify the subsequent expressions, let us choose to work near the cut-off $r = r_c$. This is similar to the spirit of the previous section where we eventually analyzed the system from the boundary point of view. Then to solve (3.69) near $r \sim r_c$ we can switch to following coordinate system

$$r = r_c(1 - \zeta) \quad (3.70)$$

Taylor expanding all the terms $\mathcal{J}(r), g(r), \frac{1}{r_c^n(1-\zeta)^n}$ in (3.69) about $\zeta = 0$, we obtain a power series solution for \mathcal{G} as:

$$\mathcal{G}(r) = \sum_{\alpha} \sum_{i=0}^{\infty} \frac{\tilde{a}_i^{(\alpha)}}{r_{c(\alpha)}^{4i} (1 - \zeta)^{4i}} \equiv \sum_{i=0}^{\infty} a_i \zeta^i \quad (3.71)$$

Since (3.69) is a second order differential equation, we can fix two coefficients and we choose $a_0 = a_1 = 0$. Then the rest of a_i 's are determined by equating coefficients of ζ^i on both sides of equation (3.69). The exact solutions are listed in Appendix E of [57]. Note that all a_i are proportional to g_s and in the limit $g_s \rightarrow 0$, $\mathcal{G} \rightarrow 0$.

To next order in γ we have an equation for \mathcal{H} that also depends on the solution that we got for \mathcal{G} . The equation for $\mathcal{H}(r)$ can be taken from (3.67) as:

$$\begin{aligned} \mathcal{H}''(r) + \left[\frac{g'(r)}{g(r)} + \frac{5}{r} + \mathcal{M}(r) \right] \mathcal{H}'(r) + \mathcal{J}(r)\mathcal{H}(r) = -\frac{2g'(r)}{g(r)}\mathcal{G}'(r) \\ - \left\{ \frac{g''(r)}{g(r)} + \left[\frac{5}{r} + \mathcal{M}(r) \right] \frac{g'(r)}{g(r)} \right\} [1 + \mathcal{G}(r)] \end{aligned} \quad (3.72)$$

To solve this we make the coordinate transformation (3.70) and plug in the series solution for $\mathcal{G}(r)$ given above. The final result for \mathcal{H} can again be expressed as a series solution in ζ in the following way:

$$\mathcal{H}(r) = \sum_{\alpha} \sum_{i=0}^{\infty} \frac{\tilde{b}_i^{(\alpha)}}{r_{c(\alpha)}^{4i} (1 - \zeta)^{4i}} \equiv \sum_{i=0}^{\infty} b_i \zeta^i \quad (3.73)$$

We again set $b_0 = b_1 = 0$ and following similar ideas used to solve for \mathcal{G} , we determine all b'_i s by equating coefficients in (3.72). The exact solution is given in Appendix E of [57]. Again note that all b_i are of at least $\mathcal{O}(g_s)$ and thus with $g_s \rightarrow 0$, $\mathcal{H} \rightarrow 0$.

Finally the second order in γ is a much more involved equation that uses results of the previous two equations to determine \mathcal{K} . This is given by:

$$\begin{aligned} \mathcal{K}''(r) + \left(\frac{g'(r)}{g(r)} + \frac{5}{r} + \mathcal{M}(r) \right) \mathcal{K}'(r) + \mathcal{J}(r) \mathcal{K}(r) = - \frac{2g'(r)}{g(r)} \mathcal{H}'(r) \quad (3.74) \\ - \left[\frac{g''(r)}{g(r)} + \left(\frac{5}{r} + \mathcal{M}(r) \right) \frac{g'(r)}{g(r)} \right] \mathcal{H}(r) - \left(\kappa_0 + \frac{g'^2(r)}{g^2(r)} \right) [1 + \mathcal{G}(r)] \end{aligned}$$

which could also be solved using another series expansion in ζ^i (we haven't attempted it here). Therefore combining (3.71) and (3.73) we finally have the solution for the metric perturbation:

$$\tilde{\phi}(r, |\omega|)_{\pm} = g(r)^{\pm i \frac{|\omega|}{4\pi T_c}} \left[1 + \mathcal{G}(r) \pm i \frac{|\omega|}{4\pi T_c} \mathcal{H}(r) - \frac{|\omega|^2}{16\pi^2 T_c^2} \mathcal{K}(r) + \dots \right] \quad (3.75)$$

We can analyze this in the regime where the gravitons have very small energy, i.e $\omega \rightarrow 0$ or equivalently $\gamma \rightarrow 0$. In this limit we can Taylor expand $\tilde{\phi}(r, |\omega|)$ about $\gamma = 0$ to give us the two possible solutions:

$$\begin{aligned} \tilde{\phi}(r, |\omega|)_{\pm} = & 1 + \mathcal{G}(r) \pm i \frac{|\omega|}{4\pi T_c} \left\{ \mathcal{H}(r) + [1 + \mathcal{G}(r)] \log g(r) \right\} \quad (3.76) \\ & - \frac{|\omega|^2}{16\pi^2 T_c^2} \left\{ \mathcal{K}(r) + \mathcal{H}(r) \log g(r) + [1 + \mathcal{G}(r)] \log^2 g(r) \right\} + \mathcal{O}(|\omega|^3) \end{aligned}$$

which consequently means that to the first order in ω the off diagonal gravitational perturbation at low energy is given by two possible solutions corresponding to positive and negative frequencies as:

$$\phi(r, \omega)_{\pm} = [1 + \mathcal{G}(r)] \bar{\phi}(\omega) \pm i \frac{|\omega|}{4\pi T_c} \left\{ \mathcal{H}(r) + [1 + \mathcal{G}(r)] \log g(r) \right\} \bar{\phi}(\omega) \quad (3.77)$$

As is well known, following [142] [135], we can define field on the right **R** and left **L** quadrant of the Kruskal plane in terms of $\phi_+(r, \omega)$ and $\phi_-(r, \omega)$ in the following way:

$$\begin{aligned} \phi_{\mathbf{R}, \pm}(\omega, r) &= \phi_{\pm}(\omega, r) \quad \text{in } \mathbf{R} \\ &= 0 \quad \text{in } \mathbf{L} \\ \phi_{\mathbf{L}, \pm}(\omega, r) &= \phi_{\pm}(\omega, r) \quad \text{in } \mathbf{L} \\ &= 0 \quad \text{in } \mathbf{R} \end{aligned} \quad (3.78)$$

Now $\phi_{\mathbf{R},\pm}, \phi_{\mathbf{L},\pm}$ contain positive and negative frequency modes but a certain linear combination of $\phi_{\mathbf{R},\pm}, \phi_{\mathbf{L},\pm}$ gives purely positive or purely negative frequency modes in the entire Kruskal plane [142] [135]. Furthermore imposing that positive frequency modes are in falling at the horizon in \mathbf{R} quadrant and negative frequency modes are outgoing at the horizon in \mathbf{R} fixes two combinations :

$$\begin{aligned}\phi_{\text{pos}} &= e^{\omega/T_c} \phi_{\mathbf{R},-}(\omega, r) + e^{\omega/2T_c} \phi_{\mathbf{L},-}(\omega, r) \\ \phi_{\text{neg}} &= \phi_{\mathbf{R},+}(\omega, r) + e^{\omega/2T_c} \phi_{\mathbf{L},+}(\omega, r)\end{aligned}\quad (3.79)$$

With (3.79) we see that we can define fields in $\mathbf{R}(\mathbf{L})$ quadrant as linear combination of positive and negative frequency modes

$$\begin{aligned}\phi_{\mathbf{R}}(\omega, r) &\equiv \tilde{a}_0[\phi_{\mathbf{R},+}(\omega, r) - e^{\omega/T_c} \phi_{\mathbf{R},-}(\omega, r)] \equiv \phi_1 \\ \phi_{\mathbf{L}}(\omega, r) &\equiv \tilde{a}_0 e^{\omega/2T_c} [\phi_{\mathbf{L},+}(\omega, r) - \phi_{\mathbf{L},-}(\omega, r)] \equiv \phi_2\end{aligned}\quad (3.80)$$

where we have identified $\phi_{\mathbf{R}}(\phi_{\mathbf{L}})$ with the thermal field $\phi_1(\phi_2)$ defined on the complex time contour which familiarly appears in the Schwinger-Keldysh propagators of real time thermal field theory. Here \tilde{a}_0 is a constant. The final physical quantity that we will extract from here will only depend on \mathcal{T} , as we will show soon.

Having got the graviton fluctuations $\phi(r, \omega) \equiv \phi_{\mathbf{R}}(\omega, r)$, we are almost there to compute the viscosity η using (3.57). Our next step would be to compute the Schwinger-Keldysh propagator $G_{11}^{\text{SK}}(0, \vec{q})$. All we now need is to write the action (3.59) as (3.58) and from there extract the Schwinger-Keldysh propagator. This analysis is similar to the one that we did in the previous section, so we could be brief (see also [120]). The action (3.59) can be used to get the boundary action once we shift $\phi(r, \omega)$ to $\phi(r, \omega) + \delta\phi(r, \omega)$ in the following way:

$$\begin{aligned}S_{\text{SG}}^{(2)}(\phi + \delta\phi) &= \frac{g(r_c)^{-1/2}}{8\pi G_N} \int \frac{d\omega d^3q}{(2\pi)^4} \int_{r_h}^{r_c} dr \left\{ A(r) \phi(r, -\omega) \phi''(r, \omega) + B(r) \phi'(r, -\omega) \phi'(r, \omega) \right. \\ &\quad + C(r) \phi(r, -\omega) \phi'(r, \omega) + D(r) \phi(r, -\omega) \phi(r, \omega) + [2A(r) \phi''(r, \omega) \\ &\quad - 2B(r) \phi''(r, \omega) - 2B'(r) \phi'(r, \omega) - C'(r) \phi(r, \omega) + 2D(r) \phi(r, \omega) \\ &\quad + A''(r) \phi(r, \omega) + 2A'(r) \phi'(r, \omega)] \delta\phi(r, -\omega) + \partial_r [2B(r) \phi'(r, \omega) \delta\phi(r, -\omega) \\ &\quad \left. + C(r) \phi(r, \omega) \delta\phi(r, -\omega) + A(r) \phi(r, \omega) \delta\phi'(r, -\omega) - \partial_r (A(r) \phi(r, \omega)) \delta\phi(r, -\omega) \right\}\end{aligned}\quad (3.81)$$

Plugging in the background value of $\phi(r, \omega)$ will tell us that only the boundary term survives. And as before, to cancel the $A(r)\phi(r, \omega)\delta\phi'(r, -\omega)$ we will have to add the Gibbons-Hawking term to the action [61]. The net result is the following boundary action:

$$\begin{aligned} S_{\text{SG}}^{(2)} &= \frac{g(r_c)^{-1/2}}{8\pi G_N} \int \frac{d\omega d^3q}{(2\pi)^4} \phi(r, -\omega) \left\{ \frac{1}{2} [C(r) - A'(r)] + [B(r) - A(r)] \frac{\phi'(r, -\omega)}{\phi(r, -\omega)} \right\} \phi(r, \omega) \Big|_{r_h}^{r_c} \\ &\equiv \frac{1}{8\pi G_N \sqrt{g(r_c)}} \int \frac{d\omega d^3q}{(2\pi)^4} \mathcal{F}(\omega, r) \Big|_{r_h}^{r_c} \end{aligned} \quad (3.82)$$

Now comparing (3.58) with 3.82 we see that the terms between the braces combine to give us the required Schwinger-Keldysh propagator:

$$\begin{aligned} G_{11}^{\text{SK}}(0, \vec{q}) &= \lim_{\omega \rightarrow 0} \frac{1}{4\pi G_N \sqrt{g(r_c)}} \frac{\mathcal{F}(\omega, r)}{\phi_1(r, \omega) \phi_1(r, -\omega)} \Big|_{r_h}^{r_c} \\ &= \lim_{\omega \rightarrow 0} \frac{1}{4\pi G_N \sqrt{g(r_c)}} \left\{ \frac{1}{2} [C(r) - A'(r)] + [B(r) - A(r)] \frac{\phi_1'(r, -\omega)}{\phi_1(r, -\omega)} \right\} \Big|_{r_h}^{r_c} \end{aligned} \quad (3.83)$$

where we assume¹¹ that $\phi_1(r_h, \omega) = \tilde{a}_0[\phi_{\mathbf{R},+}(r_h, \omega) - e^{\omega/T_c} \phi_{\mathbf{R},-}(r_h, \omega)]$. Now to evaluate the shear viscosity from the above result we need to perform two more steps:

- Evaluate the contributions from the UV cap that we attach from $r = r_c$ to $r = \infty$.
- Take the imaginary part of the resulting *total* Schwinger-Keldysh propagator. This should give us result independent of the cut-off.

To evaluate the first step i.e contributions from the UV cap, we need to see precisely the singularity structure of $S_{\text{SG}}^{(2)}$. The second step would then be to extract the imaginary part of SK propagator from there. Since the imaginary part can only come from the second term of (3.82), we only need to evaluate:

$$\lim_{\omega \rightarrow 0} \frac{1}{4\pi G_N \sqrt{g(r_c)}} [B(r) - A(r)] \frac{\phi_1'(r, -\omega)}{\phi_1(r, -\omega)} \Big|_{r_c}^{\infty} \quad (3.84)$$

¹¹At this point one might worry that the solution for ϕ_1 is only known around r_c . That this is not the case can be seen in the following way: Integration by parts gives (3.83) which says one only needs to know the value of the field ϕ_1 at r_c and r_h . The solution for $\tilde{\phi}_1 = \frac{\phi_1}{\phi_1}$ is given in (3.75) from which it is clear that $\phi_1(r_h) = 0$ as $g(r_h) = 0$. Furthermore to know η we only need to know the imaginary part of (3.83), which is evaluated using (3.85) in (3.83) and using boundary values of $\phi_1(r_c)$ and $\phi_1(r_h)$.

with $\phi(r, -\omega)$ being the graviton fluctuation in the regime $r > r_c$. To analyze this let us first consider a case where $g_s \rightarrow 0$ and $(\mathcal{G}(r), \mathcal{H}(r), \mathcal{K}(r), \dots) \rightarrow 0$. In this limit we expect for $r_h \leq r \leq r_c$:

$$\begin{aligned} B(r) - A(r) &= -\frac{1}{2g_s^2} g(r)r^5 + \mathcal{O}(g_s N_f) \\ \phi_1(r, \omega) &= \tilde{a}_0 \left[-\frac{\omega}{T_c} \left(1 + \mathcal{G} - i\frac{|\omega|}{4\pi T_c} \mathcal{H} \right) + i\frac{|\omega|}{2\pi T_c} \mathcal{H} + i\frac{|\omega|}{2\pi T_c} \log g(1 + \mathcal{G}) \right] \\ \frac{\phi'_1(r, -\omega)}{\phi_1(r, -\omega)} &= \frac{g'(r)}{g(r)} \left(\frac{2\pi}{4\pi^2 + \log^2 g(r)} \right) \end{aligned} \quad (3.85)$$

The above considerations would mean that the contribution to the viscosity, η_1 , for this simple case without incorporating the UV cap will be:

$$\eta_1 = \frac{r_h^4}{2\pi T_c g_s^2 G_N \sqrt{g(r_c)}} \left(\frac{1}{4\pi + \frac{1}{\pi} \log^2 g(r_c)} \right) = \frac{\mathcal{T}^3 L^2}{2g_s^2 G_N} \left(\frac{1}{4\pi + \frac{1}{\pi} \log^2 g(r_c)} \right) \quad (3.86)$$

where we have used the relations $\pi T_c \sqrt{g(r_c)} = \left[r_h \sqrt{\bar{h}(r_h)} \right]^{-1}$ and $\bar{h}(r_h) \approx \frac{L^4}{r_h^4}$ in this limit. This helps us to write everything in terms of \mathcal{T} and not the scale dependent temperature T_c . In fact as we show below, once we incorporate the contributions from the UV cap, the r_c dependence of the above formula will also go away and the final result will be completely independent of the cut-off. Note that in the limit $r_c \rightarrow \infty$ we recover the result for the cascading theory.

Combining all the ingredients together, the contribution to the viscosity in the limit where $(\mathcal{G}(r), \mathcal{H}(r), \mathcal{K}(r), \dots)$ etc are non-zero can now be presented succinctly as (although η_1 below doesn't have any real meaning on the gauge theory side as this is an intermediate quantity):

$$\eta_1 = \frac{r_h^5 \sqrt{\bar{h}(r_h)}}{2g_s^2 G_N} \left\{ \frac{1 + \frac{r_h^5 g(r_c)}{4r_h^4} \left[\frac{\mathcal{H}'}{1+\mathcal{G}} - \frac{\mathcal{H}\mathcal{G}'}{(1+\mathcal{G})^2} \right]}{4\pi + \frac{1}{\pi} \left[\log g(r_c) + \frac{\mathcal{H}}{1+\mathcal{G}} \right]^2} \right\} \quad (3.87)$$

Note that the above expression is exact for our background at least in the limit where we take the leading order r^5 singularity of the background. This is motivated from our detailed discussion that we gave in the previous section. Note that the second term in the action (3.82) is exactly the second equation of the set (2.69) whose

singularity structure has been shown to be renormalizable. Thus taking the leading order singularity r^5 instead of the actual $r_{(\alpha)}^5$ will not change anything if we carefully compensate the coefficients with appropriate $g_s N_f, g_s M^2/N$ factors!

But this is still not the complete expression as we haven't added the contributions from the UV cap. Before we do that, we want to re-address the singularity structure of the above expression. The worrisome aspect is the existence of r_c^5 factor in (3.87). Does that create a problem for our case?

The answer turns out to be miraculously no, because of the form of \mathcal{H} and \mathcal{G} given in (3.73) and (3.71). This, taking only the leading powers of r_c , yields:

$$\mathcal{H}' = -\frac{4\tilde{b}_1}{r_c^5} - \frac{8\tilde{b}_2}{r_c^9} + \dots, \quad \mathcal{G}' = -\frac{4\tilde{a}_1}{r_c^5} - \frac{8\tilde{a}_2}{r_c^9} + \dots \quad (3.88)$$

killing the r_c^5 dependence in (3.87)¹². This would make η_1 completely finite and all the r_c dependences would go as $\mathcal{O}(1/r_c)$. Therefore we expect the contribution to the viscosity from the UV cap to go like:

$$\eta_2 \equiv \eta|_{r_c}^\infty = \sum_{i=0}^{\infty} \frac{G_i}{r_c^{4i}} \quad (3.89)$$

where the total viscosity will be defined as $\eta \equiv \eta_1 + \eta_2$. As this is a physical quantity we expect it to be independent of the scale. Therefore

$$\frac{\partial \eta}{\partial r_c} = 0 \quad (3.90)$$

which will give us similar Callan-Symanzik type equations, as discussed in the previous section, from where we could derive the precise forms for G_i in 3.89. Finally when the dust settles, the result for shear viscosity can be expressed as:

$$\eta = \frac{\mathcal{T}^5 \sqrt{\bar{h}(\mathcal{T})}}{2g_s^2 G_N} \left[\frac{1 + \sum_{k=1}^{\infty} \alpha_k e^{-4k\mathcal{N}_{uv}}}{4\pi + \frac{1}{\pi} \log^2(1 - \mathcal{T}^4 e^{-4\mathcal{N}_{uv}})} \right] \quad (3.91)$$

where α_k are functions of \mathcal{T} that can be easily determined from the coefficients (\bar{a}_i, \bar{b}_i) in (3.71) and (3.73) or (a_i, b_i) worked out in Appendix E of [57]; and $\bar{h}(\mathcal{T}) \equiv$

¹²It is now easy to see why $\tilde{b}_1 = \tilde{a}_1 = 0$ is consistent. For non-zero \tilde{b}_1, \tilde{a}_1 there would have been additional $\log r$ terms from $r_{(\alpha)}^5$. These would have made the theory non-renormalizable. Thus holographic renormalizability would demand $\tilde{b}_1 = \tilde{a}_1 = 0$ from the very beginning – consistent with what we choose earlier.

$\frac{L^4}{\mathcal{T}^4} + \mathcal{O}(g_s, N_f, M)$. Observe that the final result for shear viscosity is completely independent of r_c and T_c ; and only depend on \mathcal{T} and the degrees of freedom at the UV i.e through $e^{-\mathcal{N}_{uv}}$. Needless to say, for large enough \mathcal{N}_{uv} (which is always the case for our case because $\mathcal{N}_{uv} \rightarrow \epsilon^{-n}, n \geq 1$), the shear viscosity is only sensitive to the characteristic temperature \mathcal{T} of the cascading theory. The interesting thing however is that the shear viscosity with finite but large enough \mathcal{N}_{uv} can be *smaller* than or *equal to* the shear viscosity with $\mathcal{N}_{uv} \rightarrow \epsilon^{-n}, n \gg 1$ i.e for the parent cascading theory provided:

$$\alpha_k \leq \frac{1}{4\pi^2} \sum_{n \in \mathbf{Z}} \frac{\mathcal{T}^{4k}}{n(k-n)}, \quad n \leq k, \quad k \in \mathbf{Z} \quad (3.92)$$

in the limit of small characteristic temperature \mathcal{T} . This will have effect on the viscosity to entropy ratio, to which we turn next.

3.2.2 η/s

Going back to the stress tensor of fluid described by hydrodynamics, one can obtain [57]

$$\langle \delta T_{ij} \rangle = -\frac{\eta}{\varepsilon + P} \left(\nabla_i \langle T_j^0 \rangle + \nabla_j \langle T_i^0 \rangle - \frac{2}{3} \delta_{ij} \nabla_l \langle T^{l0} \rangle \right) - \frac{\zeta}{\varepsilon + P} \delta_{ij} \nabla_l \langle T^{l0} \rangle \quad (3.93)$$

where δT^{ij} is the deviation from the ideal fluid stress tensor T^{ij} in the fluid rest frame and ε and P are the local energy density and the pressure, respectively. Using the thermodynamic identity $Ts = \varepsilon + P$ where s is the entropy density, the two coefficients can be also written as η/Ts and ζ/Ts . Since the temperature is the only relevant energy scale in the highly relativistic fluid, one can easily see that the importance of the viscous terms depends on the size of the dimensionless ratios η/s and ζ/s . Thus not the absolute value of shear viscosity rather the ratio η/s is what is relevant in viscous hydrodynamics.

Furthermore, the dimensionless number η/s has been shown to be universal for a large class of gauge theories where the dual graviton perturbation is a scalar field [143] and even more strikingly $\eta/s = 1/4\pi$ has been conjectured to be its lowest possible value [144] for *any* fluid using arguments of [145]. Although it has been shown by

[57][146]-[154] that the lower bound can be violated for gauge theories which have higher derivative terms (i.e. curvature square terms and beyond) in their dual gravity, these theories may violate other constraints and may not be most relevant for physics of fluids. In any case, the ratio η/s becomes crucial in describing fluid dynamics and in this section we will compute this ratio using dual gravity.

We will calculate the ratio for two cases i.e one with only RG flow, and the other with both RG flow and curvature squared corrections. As usual the former is easier to handle so we discuss this first.

Starting with the type IIB supergravity action (2.19) in ten dimension, the entropy is given by Wald's formula [155],[156],[157],[158]

$$\mathcal{S} = -2\pi \oint dx dy dz d^5 \mathcal{M} \sqrt{\mathcal{P}} \frac{\partial \mathcal{L}_{10}}{\partial R_{abcd}} \epsilon^{ab} \epsilon^{cd} \quad (3.94)$$

where the integral is over the eight dimensional surface of the horizon at $r = r_h$, \mathcal{L}_{10} is the Lagrangian density of the action in (2.19), \mathcal{P}_{ab} , $a, b = 1..8$ is the induced 8×8 metric at horizon, ϵ_{ab} is the bi normal normalized to $\epsilon_{ab} \epsilon^{ab} = -2$. Finally using explicit expression for the metric (2.91) with warp factor given by (2.31) near horizon, we have

$$\begin{aligned} s &= \frac{\mathcal{S}}{V_3} = -\frac{\pi r_h^5}{108 V_3 \kappa_{10}^2} \oint dx dy dz d^5 \mathcal{M} \sin \theta_1 \sin \theta_2 \sqrt{h(r_h, \theta_1, \theta_2)} \frac{\partial \mathcal{L}_{10}}{\partial R_{abcd}} \epsilon^{ab} \epsilon^{cd} \\ &= \frac{r_h^3 L^2}{2 g_s^2 G_N} \left\{ 1 + \frac{3 g_s M_{\text{eff}}^2}{2 \pi N} \left[1 + \frac{3 g_s N_f^{\text{eff}}}{2 \pi} \left(\log r_h + \frac{1}{2} \right) - \frac{g_s N_f^{\text{eff}}}{4 \pi} \right] \log r_h \right\}^{1/2} \end{aligned} \quad (3.95)$$

where V_3 is the infinite three dimensional volume and we have used the definition of five dimensional Newton's constant G_N introduced in (3.60).

Once we replace r_h by the characteristic temperature \mathcal{T} , we see that the entropy is only sensitive to the temperature and is independent of any other scale of the theory. Since the above result is also independent of \mathcal{N}_{uv} it would seem that the Wald formula only gives the entropy for the theory with $\mathcal{N}_{uv} = \infty$ i.e for the parent cascading theory¹³. The interesting question now would be to ask what is the entropy for the theory whose UV description is different from the parent cascading theory? In other words, what is the effect of the UV cap attached at $r = r_c$ on the entropy?

¹³This can be argued by observing that fact that in a renormalizable theory, like ours, the dependences on degrees of freedom go like $\mathcal{O}(e^{-\mathcal{N}_{uv}})$ corrections as we saw in the previous sections.

To evaluate this, observe first that in finite temperature gauge theory, entropy density of a thermalized medium having stress tensor $\langle T^{\mu\nu} \rangle = \text{diagonal}(\epsilon, P, P, P)$ is given by

$$s = \frac{\epsilon + P}{T} \quad (3.96)$$

where ϵ is the energy density, $P \equiv P_x = P_y = P_z$ is the pressure of the medium and T being the temperature. With our gravity dual we can compute the stress tensor $\langle T_{\text{med}}^{pq} \rangle$ (and thus the energy $\epsilon = \langle T_{\text{med}}^{00} \rangle$ and the pressure $P = \langle T_{\text{med}}^{11} \rangle$) of the medium through equation of the form (3.36), i.e

$$\langle T_{\text{med}}^{pq} \rangle = \frac{\delta_b \mathbf{S}_{\text{total}}}{\delta_b \mathbf{g}_{pq}} \quad (3.97)$$

where again $p, q = 0, 1, 2, 3$ and \mathbf{g}_{pq} is the four dimensional metric obtained from the ten dimensional OKS-BH metric $g_{ij}, i, j = 0, 1, \dots, 9$; and δ_b operation has been defined earlier. There are two ways by which we could get a four-dimensional metric from the corresponding ten-dimensional one. The first way is to integrate out the θ_i, ϕ_i directions to get the four-dimensional effective theory. This is because the warp factor for our case is dependent on the θ_i directions. The second way is to work on a slice in the internal space. The slice is coordinated by choosing some specific values for the internal angular coordinates. Such a choice is of course ambiguous, and we can only rely on it if the physical quantities that we want to extract from our theory is not very sensitive to the choice of the slice. Clearly the first way is much more robust but unfortunately not very easy to implement. We will therefore follow the second way by choosing the the five dimensional slice as $\theta_1 = \theta_2 = \pi, \psi = \phi_1 = \phi_2 = 0$ and thus obtaining

$$\mathbf{g}_{\mu\nu} \equiv g_{\mu\nu}(\theta_i = \pi, \psi = \phi_i = 0) \quad (3.98)$$

with $\mu, \nu = 0, 1, 2, 3, 4$. The next step would be to evaluate all the fluxes and the axio-dilaton on the slice. To do this we define:

$$\begin{aligned} |\mathbf{H}_3|^2 &= |H_3|^2(\theta_i = \pi, \psi = \phi_i = 0); & |\mathbf{F}_3|^2 &= |\tilde{F}_3|^2(\theta_i = \pi, \psi = \phi_i = 0) \\ |\mathbf{F}_5|^2 &= |\tilde{F}_5|^2(\theta_i = \pi, \psi = \phi_i = 0); & |\mathbf{F}_1|^2 &= |F_1|^2(\theta_i = \pi, \psi = \phi_i = 0) \\ \Phi &= \Phi(\theta_i = \pi, \psi = \phi_i = 0) \end{aligned} \quad (3.99)$$

Once the fluxes have been defined, we need the description for $\mathbf{S}_{\text{total}}$ which gives rise to (3.98). This is easily obtained from (2.19) as:

$$\begin{aligned} \mathbf{S}_{\text{total}} &= \frac{1}{2\kappa_5^2} \int d^5x e^{-2\Phi} \sqrt{-\mathbf{g}} \left(\mathbf{R} - 4\partial_i \Phi \partial^i \Phi - \frac{1}{2} |\mathbf{H}_3|^2 \right) \\ &\quad - \frac{1}{2\kappa_5^2} \int d^5x \sqrt{-\mathbf{g}} \left(|\mathbf{F}_1|^2 + |\mathbf{F}_3|^2 + \frac{1}{2} |\mathbf{F}_5|^2 \right) \end{aligned} \quad (3.100)$$

with \mathbf{R} being the Ricci-scalar for $\mathbf{g}_{\mu\nu}$ and $\mathbf{g} = \det \mathbf{g}_{\mu\nu}$. Note that in the definition for the *slice* sources $\mathbf{H}_3, \mathbf{F}_1, \mathbf{F}_3, \mathbf{F}_5$ and \mathbf{R} , we still have $g_{ij}, i, j \geq 5$ which we evaluate at $\theta_i = \pi, \psi = \phi_i = 0$, treating them simply as functions and not metric degrees of freedom.

To complete the background we need the line element. Here we will encounter some subtleties regarding the choice of the black-hole factors and the corresponding $g_s N_f$ type corrections to them. With the definition of $\mathbf{g}_{\mu\nu}$ the line element is:

$$\begin{aligned} ds^2 &= - \frac{\bar{g}_1(r)}{\sqrt{h(r, \pi, \pi)}} dt^2 + \frac{\sqrt{h(r, \pi, \pi)}}{\bar{g}_2(r)} dr^2 + \frac{1}{\sqrt{h(r, \pi, \pi)}} d\vec{x}^2 \quad (3.101) \\ \bar{g}_1(r) &= g_1(r, \theta_1 = \pi, \theta_2 = \pi) = 1 - \frac{r_h^4}{r^4} + \sum_{i,j=0}^{\infty} \alpha_{ij} \frac{\log^i(r)}{r^j} = 1 + \sum_{j,\alpha} \frac{\sigma_j^{(\alpha)}}{r_{(\alpha)}^j} \\ \bar{g}_2(r) &= g_2(r, \theta_1 = \pi, \theta_2 = \pi) = 1 - \frac{r_h^4}{r^4} + \sum_{i,j=0}^{\infty} \beta_{ij} \frac{\log^i(r)}{r^j} = 1 + \sum_{j,\alpha} \frac{\kappa_j^{(\alpha)}}{r_{(\alpha)}^j} \end{aligned}$$

where α_{ij}, β_{ij} are all of $\mathcal{O}(g_s N_f, g_s M)$ and only involve the parameters of the theory namely, r_h, L and μ from the embedding equation (2.16) and guarantees that $\frac{\alpha_{ij}}{r^j}, \frac{\beta_{ij}}{r^j}$ are dimensionless. On the other hand $\sigma_j^{(\alpha)}, \kappa_j^{(\alpha)}$ can incorporate zeroth orders in $g_s N_f$. However note that so far we have been assuming $g_1(r) \approx g_2(r) = g(r)$, ignoring their inherent θ_i dependences, and also the inequality stemming from the choices of α_{ij} and β_{ij} . This will be crucial in what follows, so we will try to keep the black hole factors unequal. These considerations do not change any of our previous results of course.

Now looking at the form of the metric, knowing the warp factor $h(r, \pi, \pi)$ and $\bar{g}_i(r)$, just like before we can expand the line element as AdS_5 line element plus $\mathcal{O}(g_s N_f, g_s M)$ corrections. We can then rewrite the line element (3.101) as:

$$ds^2 = - \frac{r^2}{L^2} [g(r) + l_1] dt^2 + \frac{\sqrt{h(r, \pi, \pi)}}{\bar{g}_2(r)} dr^2 + \frac{r^2}{L^2} (1 + l_2) d\vec{x}^2$$

$$\begin{aligned}
l_1(r) &= \sum_{i,j=0}^{\infty} \gamma_{ij} \frac{\log^i(r)}{r^j} \\
l_2(r) &= \sum_{i,j=0}^{\infty} \zeta_{ij} \frac{\log^i(r)}{r^j}
\end{aligned} \tag{3.102}$$

where again γ_{ij}, ζ_{ij} are of $\mathcal{O}(g_s N_f, g_s M)$ and we are taking $h(r, \pi, \pi) = \frac{L^4}{r^4} + \mathcal{O}(g_s N_f, g_s M)$. Such a way of writing the local line element tells us that there are two induced four-dimensional metrics at any point r along the radial direction:

$$\mathbf{g}_{pq}^{(0)} \equiv \text{diagonal}(-g(r), 1, 1, 1), \quad \mathbf{g}_{pq}^{(1)} \equiv \text{diagonal}(-l_1, l_2, l_2, l_2) \tag{3.103}$$

where we haven't shown the r^2/L^2 dependences. The reason for specifically isolating the four-dimensional part is to show that we can study the system from boundary point of view where the dynamics will be governed by our choice of the boundary degrees of freedom. It should also be clear, from four-dimensional point of view, the metric choice $\mathbf{g}_{pq}^{(0)}$ is directly related to the AdS geometry whereas the other choice $\mathbf{g}_{pq}^{(1)}$ is the deformation due to extra fluxes and seven branes.

The above decomposition also has the effect of simplifying our calculations of the energy momentum tensor $\langle T_{\text{med}}^{pq} \rangle$. We can rewrite the total energy momentum tensor as the sum of two parts, one coming from the AdS space and the other coming from the deformations, in the following way:

$$\begin{aligned}
\langle T_{\text{med}}^{pq} \rangle &= \frac{\delta_b \mathbf{S}_{\text{total}}^{[0]}}{\delta_b \mathbf{g}_{pq}^0} + \frac{\delta_b \mathbf{S}_{\text{total}}^{[1]}}{\delta_b \mathbf{g}_{pq}^1} \\
&\equiv \langle T_{\text{med}}^{pq} \rangle_{\text{AdS}} + \langle T_{\text{med}}^{pq} \rangle_{\text{def}} \\
\mathbf{S}_{\text{total}} &= \mathbf{S}_{\text{total}}^{[0]} + \mathbf{S}_{\text{total}}^{[1]}
\end{aligned} \tag{3.104}$$

where $\mathbf{S}_{\text{total}}^{[0]}$ is zeroth order in $g_s N_f, g_s M$ and $\mathbf{S}_{\text{total}}^{[1]}$ is higher order in $g_s N_f, g_s M$. Note that $\langle T_{\text{med}}^{pq} \rangle_{\text{AdS}} = \frac{\delta_b \mathbf{S}_{\text{total}}^{[0]}}{\delta_b \mathbf{g}_{pq}^0}$ is the well known AdS/CFT result obtained from the analysis of [63]-[67] in the limit $r_c \rightarrow \infty$. With the $\mathcal{O}(1/r)$ series expansion of our metric $\mathbf{g}_{00}^0 = 1 - r_h^4/r^4$, $\mathbf{g}_{11}^0 = \mathbf{g}_{22}^0 = \mathbf{g}_{33}^0 = 1$, the result at the boundary is

$$\begin{aligned}
\langle T_{\text{med}}^{00} \rangle_{\text{AdS}} &= \frac{r_h^4}{2g_s^2 G_N} = \frac{\mathcal{T}^4}{2g_s^2 G_N} \\
\langle T_{\text{med}}^{mn} \rangle_{\text{AdS}} &= 0 \quad m, n = 1, 2, 3
\end{aligned} \tag{3.105}$$

This only gives the CFT stress tensor as we evaluate the tensor on the AdS boundary at infinity, reproducing the expected first term of (3.95). How do we then evaluate the $\mathcal{O}(g_s N_f, g_s M)$ contributions from the deformed AdS part i.e the energy momentum tensor $\langle T_{\text{med}}^{pq} \rangle_{\text{def}}$ at any $r = r_c$ cut-off in the geometry?

In fact the procedure to evaluate exactly such a result has already been discussed in section 2.3. Therefore without going into any details, the final answer after integrating by parts, adding appropriate Gibbons-Hawkins terms and then using the equation of motion for $\mathbf{g}_{pq}^{[1]}$, we have

$$\begin{aligned} \mathbf{S}_{\text{total}}^{[1]} = & \frac{1}{8\pi G_N} \int \frac{d^4 q}{(2\pi)^4 \sqrt{g(r_c)}} \left\{ \left[\bar{C}_1^{mn}(r, q) - \bar{A}_1'^{mn}(r, q) \right] \Phi_m^{[1]}(r, q) \Phi_n^{[1]}(r, -q) \right. \\ & + \left[\bar{B}_1^{mn}(r, q) - \bar{A}_1^{mn}(r, q) \right] \left[\Phi_m^{[1]}(r, q) \Phi_n^{[1]}(r, -q) + \Phi_m^{[1]}(r, q) \Phi_n^{[1]}(r, -q) \right] \\ & \left. + \left(\bar{E}_1^m - \bar{F}_1'^m \right) \Phi_m^{[1]}(r, q) \right\} \Big|_{r_h}^{r_c} \end{aligned} \quad (3.106)$$

The values of the coefficients are given in Appendix F of [57]. The above form is exactly as we had before, and so all we now need is to get the mode expansion for $\Phi_m^{[1]}$. Note however that the subscript m can take only two values, namely $m = 0, 1$ as there are only two distinct fields $\mathbf{g}_{00}^{[1]}$ and $\mathbf{g}_{11}^{[1]} = \mathbf{g}_{22}^{[1]} = \mathbf{g}_{33}^{[1]}$. Therefore our proposed mode expansion is:

$$\Phi_m^{[1]} = \mathbf{g}_{mm}^{[1]} = \sum_{\alpha} \sum_{i=0}^{\infty} \frac{\mathbf{s}_{mm}^{(i)[\alpha]}}{r_{c(\alpha)}^i} \quad (3.107)$$

Just like our analysis in section 2.4, the action in (3.106) is divergent due to terms of $\mathcal{O}(r_c^4), \mathcal{O}(r_c^3)$ and hence we need to renormalize the action. The equations for renormalization are identical to the set of equations (2.71)–(2.76), and therefore we analogously subtract the counter terms to obtain the following renormalized action:

$$\begin{aligned} \mathbf{S}_{\text{ren}}^{[1]} = & \frac{1}{8\pi G_N} \int \frac{d^4 q}{(2\pi)^4} \left[1 - \frac{r_h^4}{r_c^4} \right]^{-\frac{1}{2}} \sum_{\alpha, \beta} \left\{ \left(\sum_{i=0}^{\infty} \frac{\tilde{\mathcal{A}}_{mn(i)[1]}^{(\alpha)}}{r_{(\alpha)}^i} \right) \tilde{\mathcal{G}}^{mn[1]} \Phi_m \Phi_n \right. \\ & + X[r_{(\alpha)}] + \left(\sum_{i=0}^{\infty} \frac{\tilde{\mathcal{E}}_{mn(i)[1]}^{(\alpha)}}{r_{(\alpha)}^i} \right) \tilde{\mathcal{M}}^{mn[1]} (\Phi_m \Phi_n' + \Phi_m' \Phi_n) + H_{|\alpha|}^{mn[1]} \left[s_{nn}^{(4)[\beta]} \Phi_m + s_{mm}^{(4)[\beta]} \Phi_n \right] \\ & \left. + K_{|\alpha|}^{mn[1]} \left[-4\tilde{s}_{nn}^{(4)[\beta]} \Phi_m - 4s_{mm}^{(4)[\beta]} \Phi_n + s_{nn}^{(5)[\beta]} \Phi_m' + s_{mm}^{(5)[\beta]} \Phi_n' \right] + \left(\sum_{i=0}^{\infty} \frac{\tilde{b}_{m(i)[1]}^{(\alpha)}}{r_{(\alpha)}^i} \right) \Phi_m \right\} \end{aligned} \quad (3.108)$$

where the radial coordinate is measured at the two boundaries r_h and r_c and Φ_m are independent of r as before. Note that $X[r_{(\alpha)}]$ is a function independent of Φ_m and appears for generic renormalized action.

Now the generic form for the energy momentum tensor is evident from looking at the linear terms in the above action (3.108). This is again the same as before. However now we also need the entropy from the energy-momentum tensor as in (3.96). The result for the energy-momentum tensor at $r = r_c$ is given by:

$$\begin{aligned} \langle T_{\text{med}}^{mm} \rangle_{\text{def}} \equiv & \frac{1}{8\pi G_N} \int \frac{d^4 q}{(2\pi)^4} \frac{1}{\sqrt{g(r_c)}} \sum_{\alpha, \beta} \left[(H_{|\alpha|}^{mn[1]} + H_{|\alpha|}^{nm[1]}) s_{nn}^{(4)[\beta]} - 4(K_{|\alpha|}^{mn[1]} \right. \\ & \left. + K_{|\alpha|}^{nm[1]}) s_{nn}^{(4)[\beta]} + (K_{|\alpha|}^{mn[1]} + K_{|\alpha|}^{nm[1]}) s_{nn}^{(5)[\beta]} + \left(\sum_{i=0}^{\infty} \frac{\tilde{b}_{n(i)[1]}^{(\alpha)}}{r_{c(\alpha)}^i} \right) \delta_{nm} \right] \end{aligned} \quad (3.109)$$

The explicit expressions for the coefficients listed above, namely, $H_{|\alpha|}^{mn[1]}$, $K_{|\alpha|}^{mn[1]}$, $\tilde{b}_{n(i)[1]}^{(\alpha)}$ and $s_{nn}^{(i)[1]}$ are given in Appendix F of [57].

To complete the story we need the contribution from the UV cap. This is similar to our earlier results. The final expression for the ratio of the energy-momentum tensor to the temperature takes the simple form:

$$\begin{aligned} \frac{\langle T_{\text{med}}^{mm} \rangle_{\text{def}}}{T_b} \equiv & \frac{\pi \mathcal{T} \sqrt{h(\mathcal{T})}}{8\pi G_N} \int \frac{d^4 q}{(2\pi)^4} \sum_{\alpha, \beta} \left[(H_{|\alpha|}^{mn[1]} + H_{|\alpha|}^{nm[1]}) s_{nn}^{(4)[\beta]} - 4(K_{|\alpha|}^{mn[1]} \right. \\ & \left. + K_{|\alpha|}^{nm[1]}) s_{nn}^{(4)[\beta]} + (K_{|\alpha|}^{mn[1]} + K_{|\alpha|}^{nm[1]}) s_{nn}^{(5)[\beta]} + \sum_{j=0}^{\infty} \tilde{b}_{n(j)[1]}^{(\alpha)} \delta_{nm} e^{-j\mathcal{N}_{uv}} \right] \end{aligned} \quad (3.110)$$

We would like to make a few comments here: First, observe that the final result is independent of our choice of cut-off. Secondly, in the string frame there should be a $1/g_s^2$ dependence. Finally, we can pull out a \mathcal{T}^4 term because the coefficients have an explicit r_h^4 dependences (see Appendix F of [57]). This means that both from the AdS and the deformed calculations performed above we can show that the entropy is of the form:

$$s = \frac{\mathcal{T}^5 \sqrt{h(\mathcal{T})}}{2g_s^2 G_N} \left[1 + \mathcal{O}(g_s N_f, g_s M, e^{-\mathcal{N}_{uv}}) \right] \quad (3.111)$$

where the first part is from (3.105) and the second part is from (3.110). The result for the parent cascading theory is (3.95), and so we should regard (3.111) as the

entropy for the theory with \mathcal{N}_{uv} degrees of freedom at the boundary. Of course in the limit $\mathcal{N}_{uv} \rightarrow \epsilon^{-n}, n \gg 1$ we should recover the entropy formula (3.95) for the parent theory. All in all we see that the correction due to \mathcal{N}_{uv} degrees of freedom only goes as $e^{-\mathcal{N}_{uv}}$, so in practice this is always small for the type of \mathcal{N}_{uv} that we consider here. This means that we can use the entropy for the parent cascading theory to estimate the viscosity by entropy ratio for a system with \mathcal{N}_{uv} degrees of freedom at the UV as:

$$\frac{\eta}{s} = \left[\frac{1 + \sum_{k=1}^{\infty} \alpha_k e^{-4k\mathcal{N}_{uv}}}{4\pi + \frac{1}{\pi} \log^2(1 - \mathcal{T}^4 e^{-4\mathcal{N}_{uv}})} \right] \quad (3.112)$$

where we see that the boundary entropy term (3.95) neatly cancels the \mathcal{T}^3 coefficient in the viscosity (3.91) to give us the precise bound of $\frac{1}{4\pi}$ when $\mathcal{N}_{uv} \rightarrow \infty$. Of course from our other analysis (3.111) we might expect a $\mathcal{O}(g_s N_f, g_s M, e^{-\mathcal{N}_{uv}})$ contribution that would make (3.112) saturate the celebrated bound $\frac{1}{4\pi}$ if the total entropy density factors compensate the factors coming from the viscosity. This would seem consistent with, for example, [143]¹⁴. In fact our conjecture would be for non-zero M, N_f and $\mathcal{N}_{uv} \rightarrow \epsilon^{-n}, n \gg 1$, the bound is exactly saturated¹⁵ i.e $\frac{\eta}{s} = \frac{1}{4\pi}$.

Our second and final step would be to incorporate both the RG flow as well as curvature square corrections. As we discussed before the curvature squared corrections are typically of the form $c_3 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$ with c_3 being the coefficient (3.53) that we computed before.

The crucial point here is that (see [146] where this has also been recently emphasized) in the presence of curvature squared corrections the five dimensional metric itself changes to:

$$ds^2 = \frac{-g_1(r)}{\sqrt{h(r, \pi, \pi)}} dt^2 + \frac{\sqrt{h(r, \pi, \pi)}}{g_2(r)} dr^2 + \frac{d\vec{x}^2}{\sqrt{h(r, \pi, \pi)}} \quad (3.113)$$

where the black hole factors g_i are no longer given by (2.32). They take the following

¹⁴Provided of course if we assume that α_k 's are more general now, being functions of $\mathcal{T}, g_s M, g_s N_f$. This way even for non-zero M, N_f , whenever we have $\mathcal{N}_{uv} \rightarrow \infty$ the bound is exactly $\frac{1}{4\pi}$.

¹⁵Note that any possible deviations from $\frac{1}{4\pi}$ due to (3.88) in (3.87) *cannot* happen because the underlying holographic renormalizability will make $\tilde{a}_1 = \tilde{b}_1 = 0$, as discussed earlier. Thus the bound in itself is a rather strong result.

forms:

$$\begin{aligned} g_1(r) &= 1 - \frac{r_h^4}{r^4} + \alpha + \gamma \frac{r_h^8}{r^4} + \tilde{\alpha}_{mn} \frac{\log^m r}{r^n} \\ g_2(r) &= 1 - \frac{r_h^4}{r^4} + \alpha + \gamma \frac{r_h^8}{r^4} + \tilde{\beta}_{mn} \frac{\log^m r}{r^n} \end{aligned} \quad (3.114)$$

where $\tilde{\alpha}_{mn}, \tilde{\beta}_{mn}$ are all of $\mathcal{O}(g_s M, g_s N_f)$ and can be worked out with some effort (we will not derive their explicit forms here). Similarly we could also express (3.114) in terms of inverse powers of r to have good asymptotic behavior. Observe that we can still impose $g_1 \approx g_2$ because the corrections are to $\mathcal{O}(g_s N_f, g_s M)$, although all our previous analysis have to be changed in the presence of curvature corrections because the explicit values of $g_i(r)$ have changed. We will address these issues in our future work. Finally (α, γ) are given by

$$\alpha = \frac{4c_3\kappa}{3L^2}; \quad \gamma = \frac{4c_3\kappa}{L^2} \quad (3.115)$$

At this point one might get worried that the metric perturbation on this background would become very complicated. On the contrary our analysis becomes rather simple once we ignore terms of $\mathcal{O}(c_3 g_s M, c_3 g_s N_f)$ (which is a valid approximation with $c_3 \ll 1$). In this limit the metric perturbation can be written simply as a *linear* combination of the terms proportional to c_3 in Φ which appears in [146] and our solution (3.75), (3.76) derived for RG flow. The final result is:

$$\begin{aligned} \tilde{\phi}(r, |\omega|)_{\pm, R^2} &= 1 \pm i \frac{|\omega|}{4\pi T_c} \left\{ \mathcal{H}(r) + [1 + \mathcal{G}(r)] \log g(r) + \frac{\alpha r^8 + \gamma r_h^8}{r^8 g(r)} - \alpha + 4\gamma \frac{r_h^4}{r^4} \right\} \\ &+ \mathcal{G}(r) - \frac{|\omega|^2}{16\pi^2 T_c^2} \left\{ \mathcal{K}(r) + \mathcal{H}(r) \log g(r) + [1 + \mathcal{G}(r)] \log^2 g(r) \right\} \\ &+ \mathcal{O}(|\omega|^3) + \mathcal{O}(c_3 g_s N_f) + \mathcal{O}(c_3 g_s M) \end{aligned} \quad (3.116)$$

where we have written 2.32 as;

$$\begin{aligned} g_1(r) &= g(r) + \mathcal{O}(c_3 g_s N_f) + \mathcal{O}(c_3 g_s M) \\ g_2(r) &= g(r) + \mathcal{O}(c_3 g_s N_f) + \mathcal{O}(c_3 g_s M) \end{aligned} \quad (3.117)$$

with $g(r) = 1 - \frac{r_h^4}{r^4}$ being the usual black hole factor. Of course as emphasized above, this is valid only in the limit $c_3 \ll 1$, which at least for our background seems to be the case (see 3.53).

The above corrections are not the only changes. The entropy computed earlier also gets corrected and therefore the horizon can no longer be at $r = r_h$. To evaluate the correction to entropy we again ignore the terms of $\mathcal{O}(c_3 g_s M, c_3 g_s N_f)$. In this limit the correction terms are precisely given by the analysis of [146] and are proportional to the c_3 factor (3.53) as expected. This means that the final result for η/s including all the ingredients i.e RG flows, Riemann square corrections as well as the contributions from the UV caps; is given by:

$$\begin{aligned} \frac{\eta}{s} = & \left[\frac{1 + \sum_{k=1}^{\infty} \alpha_k e^{-4k\mathcal{N}_{uv}}}{4\pi + \frac{1}{\pi} \log^2 (1 - \mathcal{T}^4 e^{-4\mathcal{N}_{uv}})} \right] \\ & - \frac{c_3 \kappa}{3L^2 (1 - \mathcal{T}^4 e^{-4\mathcal{N}_{uv}})^{3/2}} \left[\frac{B_o(4\pi^2 - \log^2 C_o) + 4\pi A_o \log C_o}{(4\pi^2 - \log^2 C_o)^2 + 16\pi^2 \log^2 C_o} \right] \end{aligned} \quad (3.118)$$

where we see two things: one, the bound is completely independent of the cut-off $r = r_c$ in the geometry, and two, the bound *decreases* in the presence of curvature square corrections even when $\mathcal{N}_{uv} \rightarrow \epsilon^{-n}$ with $n = \mathcal{O}(1)$.¹⁶ The constants appearing in 3.118 are defined as:

$$\begin{aligned} C_o &= 1 - \mathcal{T}^4 e^{-4\mathcal{N}_{uv}} \\ A_o &= -18\mathcal{T}^8 e^{-8\mathcal{N}_{uv}} + (3\mathcal{T}^8 e^{-8\mathcal{N}_{uv}} - 47\mathcal{T}^4 e^{-4\mathcal{N}_{uv}}) \log C_o + 26\mathcal{T}^4 e^{-4\mathcal{N}_{uv}} \\ &\quad + 24(1 + \mathcal{T}^2 e^{-2\mathcal{N}_{uv}}) \log C_o \\ B_o &= -88\pi\mathcal{T}^8 e^{-8\mathcal{N}_{uv}} + 48\pi\mathcal{T}^4 e^{-4\mathcal{N}_{uv}} + 48 \end{aligned} \quad (3.119)$$

This is consistent with [146], and the only violation of η/s may be entirely from the c_3 factor provided the increase in bound from the first term of (3.118) is negligible, as we discussed earlier for (3.112). This means in particular:

$$\frac{\eta}{s} = \frac{1}{4\pi} - n_b c_3 + \mathcal{O}(\mathcal{T} e^{-\mathcal{N}_{uv}}) \quad (3.120)$$

¹⁶In [159], non-relativistic systems that appear to have no lower bound were constructed. However, these are systems which necessarily require large chemical potentials and low temperature. In highly relativistic system created at high energy colliders such as the RHIC or the LHC, chemical potentials are small and temperature is high. Our discussion here assumes that the system under discussion has such properties so that the use of thermodynamic identity $\varepsilon + P = Ts$ is valid. Hence, our discussion here is in no direct conflict with the models constructed in [159].

where n_b can be extracted from (3.118). Here we will not study the subsequent implication of this result, for example whether there exists a causality violation in our theory due to the curvature corrections as in [147]-[154]. We hope to address this in our future work.

3.3 Confinement in QCD

At low temperatures only color neutral states exist and there are no free color charges i.e. quarks in QCD at temperatures below some critical value. The mechanism which may explain this confinement of quarks was first proposed by Wilson back in the early 70's [160] where the only requirement is the existence of abelian or non-abelian gauge fields which are strongly coupled to the matter fields that confine. The strong coupling calculation can be done using lattice gauge theory and it turns out that at low temperatures one can show that the free energy of a couple of charges grow linearly with the distance between them [161][162]. This means it takes infinite amount of energy to separate two color charges which are strongly coupled to gauge fields and thus only color singlet combinations of charges have finite energy at low temperatures. The linear behavior of free energy as a function of inter charge separation is referred as linear confinement. At high temperatures the free energy takes the form of Coulomb potential [161][162] and thus the interaction between the charges is negligible for large distances. This results in a deconfined phase of matter and gauge fields at sufficiently large temperatures.

The linear confinement of quarks at large separation and low temperatures is a strong coupling phenomenon and one uses lattice QCD to compute the free energy of the bound state of quarks. On the other hand at high enough temperatures, if the gauge coupling is weak, one can compute the free energy using perturbative QCD and obtains Coulombic interactions between the quarks. Whether using perturbative QCD at weak coupling or lattice QCD and effective field theory techniques at strong couplings, one finds that at high temperatures, the Coulomb potential is Debye screened [163][161].

The study of heavy quark potential gathers special attention as it is linked to a possible signal from Quark Gluon Plasma. In particular, heavy quarks are formed during early stages of a heavy ion collision as only then there exists enough energy for their formation. On the other hand right after the collision QGP is formed as only then the temperature is high enough for quarks to be deconfined. As temperature goes down, heavy quarks form bound states and as these states are very massive, they are formed at temperatures higher than deconfinement value. This means heavy quark bound states like J/ψ can coexist with QGP and can act as probes to the medium.

In particular one can study the J/ψ bound state formed in proton-proton collisions and compare with heavy ion collision where a medium is formed. The medium will screen the $c\bar{c}$ interaction making them less bound and eventually resulting in a suppression of J/ψ production. This phenomenon is known as the J/ψ suppression and considered as a signal of QGP formation [92].

In order to quantify quarkonium suppression and analyze features of plasma that causes the suppression, one has to compute free energy of J/ψ . At large couplings, lattice QCD calculations are most reliable and there has been extensive studies quantifying the potential for heavy quarks [164]-[173]. The effective potential between the quark anti-quark pairs separated by a distance d at temperature \mathcal{T} can then be expressed succinctly in terms of the free energy $F(d, \mathcal{T})$, which generically takes the following form:

$$F(d, \mathcal{T}) = \sigma d f_s(d, \mathcal{T}) - \frac{\alpha}{d} f_c(d, \mathcal{T}) \quad (3.121)$$

where σ is the string tension, α is the gauge coupling and f_c and f_s are the screening functions¹⁷ (see for example [164]-[170] and references therein). At zero temperature free energy is the potential energy of the pair. On the other hand free energy or potential energy of quarkonium is related to the Wilson loop which we will elaborate here.

¹⁷We expect the screening functions f_s, f_c to equal identity when the temperature goes to zero. This gives the zero temperature Cornell potential.

Consider the Wilson loop of a rectangular path \mathcal{C} with space like width d and time like length T . The time like paths can be thought of as world lines of pair of quarks $Q\bar{Q}$ separated by a spatial distance d . Studying the expectation value of the Wilson loop in the limit $T \rightarrow \infty$, one can show that it behaves as

$$\langle W(\mathcal{C}) \rangle \sim \exp(-TE_{Q\bar{Q}}) \quad (3.122)$$

where $E_{Q\bar{Q}}$ is the energy of the $Q\bar{Q}$ pair which we can identify with their potential energy $V_{Q\bar{Q}}(d)$ as the quarks are static. At this point we can use the principle of holography [23] [59] [174] and identify the expectation value of the Wilson loop with the exponential of the *renormalized* Nambu-Goto action,

$$\langle W(\mathcal{C}) \rangle \sim \exp(-S_{\text{NG}}^{\text{ren}}) \quad (3.123)$$

with the understanding that \mathcal{C} is now the boundary of string world sheet. Note that we are computing Wilson loop of gauge theory living on flat four dimensional space-time $x^{0,1,2,3}$. Whereas the string world sheet is embedded in curved five-dimensional manifold with coordinates $x^{0,1,2,3}$ and r . We will identify the five-dimensional manifold with Region 3 that we discussed in section 2.4. For the correspondence in (3.123) to be valid, we need the t'Hooft coupling which is the gauge coupling in the theory to be large. On the other hand as discussed before, it is in this regime of strong coupling that linear confinement is realized in gauge theories. Thus using gauge/gravity duality is most appropriate in computing quarkonium potential.

To be consistent with the recipe in [59], we need to make sure that the induced four dimensional metric at the boundary of the string world sheet \mathcal{C} is flat. For an AdS space, this is guaranteed as long as the world sheet ends on boundary of AdS space where the induced four dimensional metric can indeed be written as $\eta_{\mu\nu}$. Using the geometry constructed in section 2.4, with metric of the form (2.91) and warp factor given by (2.120) for large r , we see that the metric is asymptotically AdS and therefore induces a flat Minkowski metric at the boundary via:

$$\lim_{u \rightarrow 0} u^2 g_{\mu\nu} = \eta_{\mu\nu} \quad (3.124)$$

where $u = r^{-1}$ and $g_{\mu\nu}$ is the full metric (including the warp factor) in Region 3. Thus we can make the identification (3.123). Once this subtlety is resolved, comparing (3.122) and (3.123) we can read off the potential

$$V_{Q\bar{Q}} = \lim_{T \rightarrow \infty} \frac{S_{\text{NG}}^{\text{ren}}}{T} \quad (3.125)$$

Thus knowing the renormalized string world sheet action, we can compute $V_{Q\bar{Q}}$ for a strongly coupled gauge theory.

For non-zero temperature, the free energy is related to the Wilson lines $W\left(\pm\frac{d}{2}\right)$ via:

$$\exp\left[-\frac{F(d, \mathcal{T})}{\mathcal{T}}\right] = \frac{\langle W^\dagger\left(+\frac{d}{2}\right) W\left(-\frac{d}{2}\right) \rangle}{\langle W^\dagger\left(+\frac{d}{2}\right) \rangle \langle W\left(-\frac{d}{2}\right) \rangle} \quad (3.126)$$

In terms of Wilson loop, the free energy (3.121) is now related to the renormalized Nambu-Goto action for the string on a background with a black-hole¹⁸. One may also note that the theory we get is a four-dimensional theory *compactified* on a circle in Euclideanised version and not a three-dimensional theory.

3.3.1 Computing the Nambu-Goto Action: Zero Temperature

Our first attempt to compute the NG action would be to consider the zero temperature for the field theory. This means that we take the black hole factors g_i in (2.91) to be identity. The string configuration we take to compute the action is shown in Fig 3.2. Note that we are considering the case when the string is exclusively in region 3 of geometry shown in Fig 2.17 and the justification for this will be given shortly. Even if the string enters region 2 and 1, the arguments for linear confinement still

¹⁸There is a big literature on the subject where quark anti-quark potential has been computed using various different approaches like pNRQCD [171]–[173], hard wall AdS/CFT [175]–[179] and other techniques [180]–[52]. Its reassuring to note that the results that we get using our newly constructed background matches very well with the results presented in the above references. This tells us that despite the large N nature there is an underlying universal behavior of the confining potential.

holds provided the warp factor is monotonic and satisfies a certain equation. We will discuss it later in this section. For the time being observe that the configuration in Fig 3.2 has one distinct advantage over all other configurations studied in the literature, namely, that because of the absence of three-forms in region 3, we will not have the UV divergence of the Wilson loop due to the logarithmically varying B field [72].

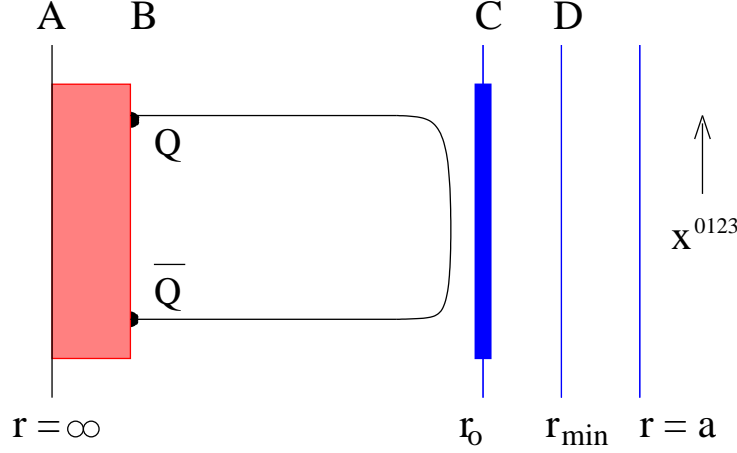


Figure 3.2: The string configuration that we will use to evaluate the Wilson loop in the dual gauge theory. The line A determines the actual boundary, with the line B denoting the extent of the seven brane. We will assume that line B is very close to the line A . The line C at $r = r_0$ denotes the boundary between Region 3 and Region 2. Region 2 is the interpolating region that ends at $r = r_{\min}$. At the far IR the geometry is cut-off at $r = a$ from the blown-up S^3 .

As the system is not dynamical, the world line for the static $Q\bar{Q}$ can be chosen to be

$$x^1 = \pm \frac{d}{2}, \quad x^2 = x^3 = 0 \quad (3.127)$$

and using $u \equiv 1/r$ we can rewrite the metric in region 3 as¹⁹:

$$\begin{aligned} ds^2 &= g_{\mu\nu} dX^\mu dX^\nu = \mathcal{A}_n(\psi, \theta_i, \phi_i) u^{n-2} \left[-g(u) dt^2 + d\vec{x}^2 \right] \\ &+ \frac{\mathcal{B}_l(\psi, \theta_i, \phi_i) u^l}{\mathcal{A}_m(\psi, \theta_i, \phi_i) u^{m+2} g(u)} du^2 + \frac{1}{\mathcal{A}_n(\psi, \theta_i, \phi_i) u^n} ds_{\mathcal{M}_5}^2 \end{aligned} \quad (3.128)$$

¹⁹We use the Einstein summation convention henceforth unless mentioned otherwise.

where \mathcal{A}_n are the coefficients that can be extracted from the a_i in (2.120), the black hole factor $g(u) = 1$ for the zero-temperature case, and $ds_{\mathcal{M}_5}^2$ is the metric of the internal space that includes the corrections given in (2.117). This can be made precise as

$$\frac{1}{\sqrt{h}} = \frac{1}{L^2 u^2 \sqrt{a_i u^i}} \equiv \mathcal{A}_n u^{n-2} = \frac{1}{L^2 u^2} \left[a_0 - \frac{a_1 u}{2} + \left(\frac{3a_1^2}{8a_0} - \frac{a_2}{2} \right) u^2 + \dots \right] \quad (3.129)$$

giving $\mathcal{A}_0 = \frac{a_0}{L^2}$, $\mathcal{A}_1 = -\frac{a_1}{2L^2}$, $\mathcal{A}_2 = \frac{1}{L^2} \left(\frac{3a_1^2}{8a_0} - \frac{a_2}{2} \right)$ and so on. Observe that since a_i , $i \geq 1$ are of $\mathcal{O}(g_s N_f)$ and $L^2 \propto \sqrt{g_s N}$, all \mathcal{A}_i are very small. The r^{-n} corrections along the radial direction given in (2.115) are accommodated above through $\mathcal{B}_l u^l$ series.

Now suppose $X^\mu : (\sigma, \tau) \rightarrow (x^{0123}, u, \psi, \phi_i, \theta_i)$ is a mapping from string world sheet to space-time. Choosing a parametrization $\tau = x^0 \equiv t$, $\sigma = x^1 \equiv x$ with the boundary of the world sheet overlapping with the world line of the $Q\bar{Q}$ pair, we see that we can have

$$\begin{aligned} X^0 &= t, & X^1 &= x, & X^2 &= X^3 = 0, & X^7 &= u(x), & X^6 &= \psi = 0 \\ (X^4, X^5) &= (\theta_1, \phi_1) = (\pi/2, 0), & (X^8, X^9) &= (\theta_2, \phi_2) = (\pi/2, 0) \end{aligned} \quad (3.130)$$

which is almost like the slice that we chose in [57]. The advantage of such a choice is to get rid of the awkward angular variables that appear for our background geometry so that we will have only a r (or u) dependent background. We also impose the boundary condition

$$u(\pm d/2) = u_\gamma \approx 0 \quad (3.131)$$

where u_γ denote the position of the seven brane *closest* to the boundary $u = 0$. The Nambu-Goto action for the string connected to this seven brane is:

$$\begin{aligned} S_{\text{string}} &= \frac{1}{2\pi\alpha'} \int d\sigma d\tau \left[\sqrt{-\det [(g_{\mu\nu} + \partial_\mu \phi \partial_\nu \phi) \partial_a X^\mu \partial_b X^\nu]} + \frac{1}{2} \epsilon^{ab} B_{ab} + J(\phi) \right. \\ &\quad \left. + \epsilon^{ab} \partial_a X^m \partial_b X^n \bar{\Theta} \Gamma_m \Gamma^{abc\dots} \Gamma_n \Theta F_{abc\dots} + \mathcal{O}(\Theta^4) \right] \end{aligned} \quad (3.132)$$

where $a, b = 1, 2$, $\partial_1 \equiv \frac{\partial}{\partial \tau}$, $\partial_2 \equiv \frac{\partial}{\partial \sigma}$. The other fields appearing in the action are the pull backs of the NS B field B_{ab} , the dilaton coupling $J(\phi)$ and the RR field strengths $F_{abc\dots}$. Its clear that if we switch off the fermions i.e $\Theta = \bar{\Theta} = 0$ the RR fields

decouple. The B_{NS} field do couple to the fundamental string but as we discussed before, in region 3 we don't expect to see any three-form field strengths. This is because the amount of B_{NS} that could leak out from region 2 to region 3 is:

$$B_{NS} = M\mathcal{S}[1 - f(r)] = M\mathcal{S} e^{-\alpha(r-r_0)}, \quad r > r_0 \quad (3.133)$$

where \mathcal{S} is the two-form:

$$\begin{aligned} \mathcal{S} = & g_s \left(b_1(r) \cot \frac{\theta_1}{2} d\theta_1 + b_2(r) \cot \frac{\theta_2}{2} d\theta_2 \right) \wedge e_\psi - \frac{g_s b_4(r)}{12\pi} \sin \theta_2 d\theta_2 \wedge d\phi_2 \quad (3.134) \\ & + \frac{3g_s}{4\pi} \left[\left(1 + g_s N_f - \frac{1}{r^{2g_s N_f}} + \frac{9a^2 g_s N_f}{r^2} \right) \log \left(\sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right) + b_3(r) \right] \sin \theta_1 d\theta_1 \wedge d\phi_1 \\ & - \frac{g_s}{12\pi} \left(2 - \frac{36a^2 g_s N_f}{r^2} + 9g_s N_f - \frac{1}{r^{16g_s N_f}} - \frac{1}{r^{2g_s N_f}} + \frac{9a^2 g_s N_f}{r^2} \right) \sin \theta_2 d\theta_2 \wedge d\phi_2 \end{aligned}$$

and b_n have been defined in section 2.4.2. We see that not only B_{NS} has an inverse r fall off, but also has a strong exponential decay as $\alpha \gg 1$. This is the main reason why there are no NS or RR three-forms in region 3, making our computation of the Wilson loop relatively easier compared to the pure Klebanov-Strassler model.

On the other hand the dilaton *will* couple *additionally* via the $J(\phi)$ term. Although this coupling of ϕ is not directly to the X^μ , we can still control this coupling by arranging the other seven-branes such that:

$$\text{Re} \left(\sum_{i=1}^{n_1} \frac{3z_i^n}{z^n} - \sum_{j=1}^{n_2} \frac{\bar{z}_j^n}{z^n} \right) < \epsilon \quad \text{for } 0 \leq n \leq m_o \quad (3.135)$$

with ϵ very small and m_o a sufficiently big number. Under this condition the dilaton will be essentially constant and the axio-dilaton τ would behave as:

$$\tau = \tau_0 + \sum_{n=1}^{\infty} \frac{\mathcal{C}_n}{r^n} + i \sum_{n>m_o}^{\infty} \frac{\mathcal{D}_n}{r^n} \quad (3.136)$$

so that its contribution to NG action can be ignored although the $\mathcal{B}_l u^l$ contribution still remains, because the seven-branes continue to affect the geometry from their energy-momentum tensors and the axion charges. In this limit both string and Einstein frame metrics are identical and the background dilaton is

$$\phi = \log g_s - g_s \mathcal{D}_{n+m_o} u^{n+m_o} + \mathcal{O}(g_s^2) \quad (3.137)$$

which, in the limit $g_s \rightarrow 0$, will be dominated by the constant term (note that m_o is fixed). On the other hand, $J(\phi) \sim \phi R_{(2)}$, where $R_{(2)}$ is the Ricci scalar for the world sheet metric. As the background space time metric run as $\mathcal{O}(1/r^n)$, we can make $R_{(2)}$ arbitrarily small by a reparametrization of the world sheet. This means we can ignore $J(\phi)$ in the string action while the NG string will see a slightly different background metric as evident from (3.132).

Thus once all the effects have been accounted for, using the metric (3.128) with the embedding X^μ given by (3.130), one can easily show that at zero temperature the NG action is given by:

$$S_{\text{NG}} = \frac{\tilde{T}}{2\pi} \int_{-\frac{d}{2}}^{+\frac{d}{2}} \frac{dx}{u^2} \sqrt{\left(\mathcal{A}_n u^n\right)^2 + \left[\mathcal{B}_m u^m + 2g_s^2 \tilde{\mathcal{D}}_{n+m_o} \tilde{\mathcal{D}}_{l+m_o} \mathcal{A}_k u^{n+l+k+2m_o} + \mathcal{O}(g_s^4)\right] \left(\frac{\partial u}{\partial x}\right)^2} \quad (3.138)$$

where we have used $\int dt = T$, $\tilde{T} = T/\alpha'$, $\tilde{\mathcal{D}}_{n+m_o} = (n+m_o)\mathcal{D}_{n+m_o}$; and $\mathcal{A}_n, \mathcal{B}_n$ and \mathcal{D}_{n+m_o} are now defined for choices of the angular coordinates given in (3.130). The above action can be condensed by redefining:

$$\mathcal{B}_m u^m + 2g_s^2 \tilde{\mathcal{D}}_{n+m_o} \tilde{\mathcal{D}}_{l+m_o} \mathcal{A}_k u^{n+l+k+2m_o} + \mathcal{O}(g_s^4) \equiv \mathcal{G}_l u^l \quad (3.139)$$

which would mean that the constraint equation i.e $\partial_1 T_1^1 = 0$, T_1^1 being the stress-tensor, for $u(x)$ derived from the action (3.138) using (3.139) can be written as

$$\frac{d}{dx} \left(\frac{(\mathcal{A}_n u^n)^2}{u^2 \sqrt{(\mathcal{A}_m u^m)^2 + \mathcal{G}_m u^m \left(\frac{\partial u}{\partial x}\right)^2}} \right) = 0 \quad (3.140)$$

implying that:

$$\frac{(\mathcal{A}_n u^n)^2}{u^2 \sqrt{(\mathcal{A}_m u^m)^2 + \mathcal{G}_m u^m u'(x)^2}} = C_o \quad (3.141)$$

where C_o is a constant, and $u'(x) \equiv \frac{\partial u}{\partial x}$. This constant C_o can be determined in the following way: as we have the endpoints of the string at $x = \pm d/2$, by symmetry the string will be U shaped and if u_{max} is the maximum value of u , we can choose the x coordinate such that $u(0) = u_{\text{max}}$ and $u'(x=0) = 0$. Plugging this in (3.141) we get:

$$C_o = \frac{\mathcal{A}_n u_{\text{max}}^n}{u_{\text{max}}^2} \quad (3.142)$$

Once we have C_o , we can use (3.140) to get the following simple differential equation:

$$\frac{du}{dx} = \pm \frac{1}{C_o \sqrt{\mathcal{G}_m u^m}} \left[\frac{(\mathcal{A}_n u^n)^4}{u^4} - C_o^2 (\mathcal{A}_m u^m)^2 \right]^{1/2} \quad (3.143)$$

which in turn can be used to write $x(u)$ as:

$$x(u) = C_o \int_{u_{\max}}^u dw \frac{w^2 \sqrt{\mathcal{G}_m w^m}}{(\mathcal{A}_n w^n)^2} \left[1 - \frac{C_o^2 w^4}{(\mathcal{A}_m w^m)^2} \right]^{-1/2} \quad (3.144)$$

where we have used $x(u_{\max}) = 0$. Now using the boundary condition given in (3.131)

i.e $x(u = u_\gamma) = d/2$, and defining $w = u_{\max} v$, $\epsilon_o = \frac{u_\gamma}{u_{\max}}$ we have

$$d = 2u_{\max} \int_{\epsilon_o}^1 dv v^2 \frac{\sqrt{\mathcal{G}_m u_{\max}^m v^m} (\mathcal{A}_n u_{\max}^n)}{(\mathcal{A}_m u_{\max}^m v^m)^2} \left[1 - v^4 \left(\frac{\mathcal{A}_n u_{\max}^n}{\mathcal{A}_m u_{\max}^m v^m} \right)^2 \right]^{-1/2} \quad (3.145)$$

At this stage we can assume all $\mathcal{A}_n > 0$. This is because for $\mathcal{A}_n > 0$ we can clearly have degrees of freedom in the gauge theory growing towards UV, which is an expected property of models with RG flows. Of course this is done to simplify the subsequent analysis. Keeping \mathcal{A}_n arbitrary will also allow us to derive the linear confinement behavior, but this case will require a more careful analysis. We will get back to this in the section **3.3.3**. Note also that similar behavior is seen for the the Klebanov-Strassler model, and we have already discussed how degrees of freedom run in regions 2 and 3. Another obvious condition is that d , which is the distance between the quarks, cannot be imaginary. From (3.145) we can see that the integral becomes complex for

$$\mathcal{F}(v) \equiv v^4 \left(\frac{\mathcal{A}_n u_{\max}^n}{\mathcal{A}_m u_{\max}^m v^m} \right)^2 > 1 \quad (3.146)$$

whereas for $\mathcal{F}(v) = 1$ the integral becomes singular. Then for d to be always real we must have

$$\mathcal{F}(v) \leq 1 \quad (3.147)$$

We will now set, without loss of generality, $\mathcal{A}_0 = 1$ and $\mathcal{A}_1 = 0$. Such a choice is of course consistent with supergravity solution for our background (as evident from (3.129)). This choice also defines our units in that we are setting the AdS throat

radius $L \equiv 1$ and this is only to make subsequent computations convenient. . Now analyzing the condition (3.147), one easily finds that we must have

$$\frac{1}{2}(m+1)\mathcal{A}_{m+3} u_{\max}^{m+3} \leq 1 \quad (3.148)$$

for d to be real. This condition puts an upper bound on u_{\max} and we can use this to constrain the fundamental string to lie completely in region 3. Observe that for AdS spaces, $\mathcal{A}_n = 0$ for $n > 0$ and hence there is no upper bound for u_{\max} . This is also the main reason why we see confinement using our background but not from the AdS backgrounds. Furthermore one might mistakenly think that generic Klebanov-Strassler background should show confinement because the space is physically cut-off due to the presence of a blown-up S^3 . Although such a scenario may imply an upper bound for u_{\max} , this doesn't naturally lead to confinement because due to the presence of logarithmically varying B_{NS} fields there are UV divergences of the Wilson loop. These divergences *cannot* be removed by simple regularization schemes [72] and creates problems in the interpretation of Wilson loop which describe the potential.

Coming back to (3.138) we see that it can be further simplified. Using (3.141), (3.142) and (3.143) in (3.138), we can write it as an integral over u :

$$\begin{aligned} S_{\text{NG}} &= \frac{\tilde{T}}{\pi} \int_{u_\gamma}^{u_{\max}} \frac{du}{u^2} \sqrt{\mathcal{G}_l u^l} \left[1 - \frac{C_o^2 u^4}{(\mathcal{A}_m u^m)^2} \right]^{-1/2} \\ &= \frac{\tilde{T}}{u_{\max} \pi} \int_{\epsilon_o}^1 \frac{dv}{v^2} \sqrt{\mathcal{G}_m u_{\max}^m v^m} \left[1 - v^4 \left(\frac{\mathcal{A}_n u_{\max}^n}{\mathcal{A}_m u_{\max}^m v^m} \right)^2 \right]^{-1/2} \end{aligned} \quad (3.149)$$

where in the second equality we have taken $v = u/u_{\max}$.

This simplified action (3.149) is however not the full story. It is also divergent in the limit $\epsilon_o \rightarrow 0$. But we can isolate the divergent part of the above integral (3.149) by first computing it as a function of ϵ_o . The result is

$$\begin{aligned} S_{\text{NG}} &\equiv S_{\text{NG}}^{\text{I}} + S_{\text{NG}}^{\text{II}} = \frac{\tilde{T}}{\pi} \frac{1}{u_{\max}} \int_{\epsilon_o}^1 \frac{dv}{v^2} \sqrt{\mathcal{G}_m u_{\max}^m v^m} \\ &+ \frac{\tilde{T}}{\pi} \frac{1}{u_{\max}} \int_{\epsilon_o}^1 \frac{dv}{v^2} \sqrt{\mathcal{G}_m u_{\max}^m v^m} \left\{ \left[1 - v^4 \left(\frac{\mathcal{A}_n u_{\max}^n}{\mathcal{A}_m u_{\max}^m v^m} \right)^2 \right]^{-1/2} - 1 \right\} \end{aligned} \quad (3.150)$$

Now by expanding $\sqrt{\mathcal{G}_m u_{\max}^m v^m} = \tilde{\mathcal{G}}_l v^l$ we can compute the first integral to be

$$S_{\text{NG}}^{\text{I}} = \frac{\tilde{T}}{\pi u_{\max}} \left(-\tilde{\mathcal{G}}_0 + \frac{\tilde{\mathcal{G}}_0}{\epsilon_o} + \sum_{l=2} \frac{\tilde{\mathcal{G}}_l}{l-1} + \mathcal{O}(\epsilon_o) + \dots \right) \quad (3.151)$$

where $\tilde{\mathcal{G}}_0 = \mathcal{G}_0$, $\tilde{\mathcal{G}}_1 = \frac{1}{2}\mathcal{G}_1 u_{\max}$ and so on. The second integral becomes

$$S_{\text{NG}}^{\text{II}} = \frac{\tilde{T}}{\pi u_{\max}} \int_0^1 \frac{dv}{v^2} \sqrt{\mathcal{G}_m u_{\max}^m v^m} \left\{ \left[1 - v^4 \left(\frac{\mathcal{A}_n u_{\max}^n}{\mathcal{A}_m u_{\max}^m v^m} \right)^2 \right]^{-1/2} - 1 \right\} + \mathcal{O}(\epsilon_o^3) \quad (3.152)$$

where the ϵ_o dependence here appears to $\mathcal{O}(\epsilon_o^3)$; and we have set $\mathcal{G}_1 = 0$ without loss of generality. Now combining the result in (3.151) and (3.152), we can obtain the renormalized action by subtracting the divergent term $\mathcal{O}(1/\epsilon)$ in the limit $\epsilon_o \rightarrow 0$ and we obtain the following result

$$\begin{aligned} S_{\text{NG}}^{\text{ren}} &= \frac{\tilde{T}}{\pi u_{\max}} \left\{ -\tilde{\mathcal{G}}_0 + \sum_{l=2} \frac{\tilde{\mathcal{G}}_l}{l-1} - \int_0^1 \frac{dv}{v^2} \sqrt{\mathcal{G}_m u_{\max}^m v^m} + \mathcal{O}(g_s^2) \right. \\ &\quad \left. + \int_0^1 \frac{dv}{v^2} \sqrt{\mathcal{G}_m u_{\max}^m v^m} \left[1 - v^4 \left(\frac{\mathcal{A}_n u_{\max}^n}{\mathcal{A}_m u_{\max}^m v^m} \right)^2 \right]^{-1/2} + \mathcal{O}(\epsilon_o) \right\} \quad (3.153) \end{aligned}$$

where the third term in (3.153), including the $\mathcal{O}(g_s^2)$ correction, is related to the action for a straight string in this background in the limit $g_s \rightarrow 0$. Our subtraction scheme is more involved because the straight string sees a complicated metric due to the background dilaton and non-Ricci flat unwarped metric. This effect is *independent* of any choice of the warp factor and we expect this action to be finite in the limit $\epsilon_o \rightarrow 0$.

Once we have the finite action, we should use this to compute the $Q\bar{Q}$ potential through (3.125). Looking at (3.145) we observe that the relation between d and u_{\max} is parametric and can be quite involved depending on the coefficients \mathcal{A}_n of the warp factor. If we have $\mathcal{A}_n = 0$, $\mathcal{G}_n = 0$ for $n > 0$, we recover the well known AdS result, namely: $d \sim u_{\max}$ and $V_{Q\bar{Q}} \sim \frac{1}{d}$. But in general (3.145) and (3.153) should be solved together to obtain the potential.

As it stands, (3.145) and (3.153) are both rather involved. So to find some correlation between them, we will observe the limiting behavior of u_{\max} . Therefore in the

following, we will study the behavior of d and $S_{\text{NG}}^{\text{ren}}$ for the cases where u_{max} is large and small.

Quark-Anti quark potential for small u_{max}

Let us first consider the case where u_{max} is small. In this limit we can ignore higher order terms in u_{max} and approximate

$$\mathcal{A}_n u_{\text{max}}^n = \mathcal{A}_0 + \mathcal{A}_2 u_{\text{max}}^2 \equiv 1 + \eta \quad (3.154)$$

where $\mathcal{A}_0 = 1$ and $\mathcal{A}_2 u_{\text{max}}^2 = \eta$. Using this we can write both (3.145) and (3.153) as Taylor series in η around $\eta = 0$. The result is

$$\begin{aligned} d &= \sqrt{\eta} \left[a_0 + a_1 \eta + \mathcal{O}(\eta^2) \right] \\ S_{\text{NG}}^{\text{ren}} &= \frac{\tilde{T}}{\pi} \left[\frac{b_0 + b_1 \eta + \mathcal{O}(\eta^2)}{\sqrt{\eta}} \right] \end{aligned} \quad (3.155)$$

with a_0, a_1, b_0, b_1 are defined in the following way:

$$\begin{aligned} a_0 &= \frac{2}{\sqrt{\mathcal{A}_2}} \int_0^1 dv \frac{v^2}{\sqrt{1-v^4}} = \frac{1.1981}{\sqrt{\mathcal{A}_2}} \\ a_1 &= \frac{2}{\sqrt{\mathcal{A}_2}} \int_0^1 dv \frac{v^2}{\sqrt{1-v^4}} \left[\frac{1-v^6}{1-v^4} + \left(\frac{\mathcal{G}_2 - 4\mathcal{A}_2}{2\mathcal{A}_2} \right) v^2 \right] \\ b_0 &= \sqrt{\mathcal{A}_2} \left[-1 + \int_0^1 dv \left(\frac{1 - \sqrt{1-v^4}}{v^2 \sqrt{1-v^4}} \right) \right] = -0.62 \sqrt{\mathcal{A}_2} \\ b_1 &= \frac{1}{2\sqrt{\mathcal{A}_2}} \left\{ \mathcal{G}_2 + \int_0^1 dv \left[\frac{2\mathcal{A}_2 v^4 + \mathcal{G}_2 v^2 (1+v^2)(1 - \sqrt{1-v^4})}{v^2 (1+v^2) \sqrt{1-v^4}} \right] \right\} \end{aligned} \quad (3.156)$$

where we have taken $\mathcal{G}_0 = 1$ and $\mathcal{G}_1 = 0$ without loss of generality. In this limit clearly increasing η increases d , the distance between the quarks. For small η , $d = a_0 \sqrt{\eta}$, and therefore the Nambu-Goto action will become:

$$S_{\text{NG}}^{\text{ren}} = \tilde{T} \left[- \left(\frac{a_0 |b_0|}{\pi} \right) \frac{1}{d} + \left(\frac{b_1}{\pi a_0} \right) d + \mathcal{O}(d^3) \right] \quad (3.157)$$

where all the constants have been defined in (3.156). Using (3.125) we can determine the short-distance potential to be :

$$\begin{aligned} V_{Q\bar{Q}} &= \frac{1}{\alpha'} \left[- \left(\frac{a_0 |b_0|}{\pi} \right) \frac{1}{d} + \left(\frac{b_1}{\pi a_0} \right) d + \mathcal{O}(d^3) \right] \\ &= \sqrt{g_s N} \left[- \frac{0.236}{d} + (0.174 \mathcal{G}_2 + 0.095 \mathcal{A}_2) d + \mathcal{O}(d^3) \right] \end{aligned} \quad (3.158)$$

where N is the number of D3 branes in the gauge theory and we have used string tension $1/\alpha' = \sqrt{g_s N}/L^2 \equiv \sqrt{g_s N}$ as $L^2 \equiv 1$ according to our choice of units. The potential is dominated by the inverse d behavior, i.e the expected Coulombic behavior. Note that the coefficient of the Coulomb term which is a dimensionless number only depends on the number of D3 branes. Thus it is independent of the warp factor and hence should be universal. This result, in appropriate units, is of the same order of magnitude as the real Coulombic term obtained by comparing with Charmonium spectra as first modelled by [184] and subsequently by several authors [164]-[170],[171]-[173],[176]-[179]. This prediction, along with the overall minus sign, should be regarded as a success of our model (see also [185] where somewhat similar results have been derived in a string theory inspired model). The second term on the other hand is model dependent, and vanishes in the pure AdS background.

Note also that the above computations are valid for infinitely massive quark-anti quark pair. For lighter quarks, we expect the results to differ. It would be interesting to compare these results with the ones where quarks are much lighter.

Quark-Anti quark potential for large u_{\max}

Lets analyze the integrals (3.145) and (3.153) in the limit u_{\max} is close to its upper bound set by (3.148) (see also [185]). In particular if \mathbf{u}_{\max} is the upper bound of u_{\max} , then it is found by solving

$$\frac{1}{2}(m+1)\mathcal{A}_{m+3}\mathbf{u}_{\max}^{m+3} = 1 \quad (3.159)$$

We observe that both the integrals (3.145) and (3.153) are dominated by $v \sim 1$ behavior of the integrands. Near $v = 1$ and $u_{\max} \rightarrow \mathbf{u}_{\max}$ the distance d between the quark and the anti quark can be written as:

$$\begin{aligned} d &= 2 \frac{\sqrt{\mathcal{G}_m \mathbf{u}_{\max}^m \mathbf{u}_{\max}}}{\mathcal{A}_n \mathbf{u}_{\max}^n} \int_0^1 \frac{dv}{\sqrt{\mathbf{A}(1-v) + \mathbf{B}(1-v)^2}} \\ &= -2 \frac{\sqrt{\mathcal{G}_m \mathbf{u}_{\max}^m \mathbf{u}_{\max}}}{\mathcal{A}_n \mathbf{u}_{\max}^n} \left[\frac{\log \mathbf{A} - \log \left(2\sqrt{\mathbf{B}(\mathbf{A} + \mathbf{B})} + 2\mathbf{B} + \mathbf{A} \right)}{\sqrt{\mathbf{B}}} \right] \end{aligned} \quad (3.160)$$

where note that we have taken the lower limit to 0. This will not change any of our conclusion as we would soon see. On the other hand, the renormalized Nambu-Goto action for the string now becomes:

$$\begin{aligned}
S_{\text{NG}}^{\text{ren}} &= \frac{\tilde{T}}{\pi} \frac{\sqrt{\mathcal{G}_m \mathbf{u}_{\text{max}}^m}}{\mathbf{u}_{\text{max}}} \left[\int_0^1 \frac{dv}{\sqrt{\mathbf{A}(1-v) + \mathbf{B}(1-v)^2}} - 1 \right] - \frac{\tilde{T}}{\pi \mathbf{u}_{\text{max}}} + \mathcal{O}(\mathbf{u}_{\text{max}}^2) \\
&= -\frac{\tilde{T}}{\pi} \frac{\sqrt{\mathcal{G}_m \mathbf{u}_{\text{max}}^m}}{\mathbf{u}_{\text{max}}} \left[\frac{\log \mathbf{A} - \log \left(2\sqrt{\mathbf{B}(\mathbf{A} + \mathbf{B})} + 2\mathbf{B} + \mathbf{A} \right)}{\sqrt{\mathbf{B}}} - 1 \right] \\
&\quad - \frac{\tilde{T}}{\pi \mathbf{u}_{\text{max}}} + \mathcal{O}(\mathbf{u}_{\text{max}}^2)
\end{aligned} \tag{3.161}$$

where \mathbf{A} and \mathbf{B} are defined as:

$$\begin{aligned}
\mathbf{A} &= 4 - 2 \frac{n \mathcal{A}_n u_{\text{max}}^n}{\mathcal{A}_m u_{\text{max}}^m} \\
\mathbf{B} &= 8 \frac{n \mathcal{A}_n u_{\text{max}}^n}{\mathcal{A}_m u_{\text{max}}^m} - 3 \left(\frac{n \mathcal{A}_n u_{\text{max}}^n}{\mathcal{A}_m u_{\text{max}}^m} \right)^2 + \frac{(n^2 - n) \mathcal{A}_n u_{\text{max}}^n}{\mathcal{A}_m u_{\text{max}}^m} - 6
\end{aligned} \tag{3.162}$$

Observe that in the integral (3.160) and (3.161) we have to take the limit $u_{\text{max}} \rightarrow \mathbf{u}_{\text{max}}$. So \mathbf{A}, \mathbf{B} should be evaluated in the same limit. Interestingly, comparing (3.162) to (3.159) we see that

$$\lim_{u_{\text{max}} \rightarrow \mathbf{u}_{\text{max}}} \mathbf{A} \rightarrow 0 \tag{3.163}$$

thus vanishes when computed exactly at \mathbf{u}_{max} . The other quantity \mathbf{B} remains finite at that point and in fact behaves as:

$$\mathbf{B} = \frac{n^2 \mathcal{A}_n \mathbf{u}_{\text{max}}^n}{\mathcal{A}_m \mathbf{u}_{\text{max}}^m} - 4 > 0 \tag{3.164}$$

Our above computation would mean that the distance d between the quark and the anti quark, and the Nambu-Goto action will have the following dominant behavior:

$$\begin{aligned}
d &= \lim_{\epsilon \rightarrow 0} \frac{2\sqrt{\mathcal{G}_m \mathbf{u}_{\text{max}}^m} \mathbf{u}_{\text{max}}}{\mathcal{A}_n \mathbf{u}_{\text{max}}^n} \frac{\log \epsilon}{\sqrt{\mathbf{B}}} \\
S_{\text{NG}}^{\text{ren}} &= \lim_{\epsilon \rightarrow 0} \frac{\tilde{T}}{\pi} \frac{\sqrt{\mathcal{G}_m \mathbf{u}_{\text{max}}^m}}{\mathbf{u}_{\text{max}}} \frac{\log \epsilon}{\sqrt{\mathbf{B}}}
\end{aligned} \tag{3.165}$$

which means both of them have identical logarithmic divergences. Thus the finite quantity is the *ratio* between the two terms in 3.165. This gives us:

$$\frac{S_{\text{NG}}^{\text{ren}}}{d} = \frac{\tilde{T}}{\pi} \frac{\mathcal{A}_n \mathbf{u}_{\text{max}}^n}{\mathbf{u}_{\text{max}}^2} = T \times \text{constant} \tag{3.166}$$

Now using the identity (3.125) and the above relation (3.166) we get our final result:

$$V_{Q\bar{Q}} = \left(\frac{\mathcal{A}_n \mathbf{u}_{\max}^n}{\pi \mathbf{u}_{\max}^2} \right) d/\alpha' \quad (3.167)$$

which is the required linear potential between the quark and the anti quark.

Before we end this section one comment is in order. The result for linear confinement only depends on the existence of \mathbf{u}_{\max} which comes from the constraint equation (3.159). We have constructed the background such that \mathbf{u}_{\max} lies in region 3, although a more generic case is essentially doable albeit technically challenging without necessarily revealing new physics. For example when \mathbf{u}_{\max}^{-1} is equal to the size of the blown up S^3 at the IR will require us to consider a Wilson loop that goes all the way to region 1. The analysis remains similar to what we did before except that in regions 2 and 1 we have to additionally consider B_{NS} fields of the form $u^{\epsilon(\alpha)}$ and $\log u$ respectively. Of course both the metric and the dilaton will also have non-trivial u -dependences in these regions. One good thing however is that the Wilson loop computation have no UV or IR divergences whatsoever despite the fact that now the analysis is technically more challenging. Our expectation would be to get similar linear behavior as (3.167) here too.

3.3.2 Computing the Nambu-Goto Action: Non-Zero Temperature

After studying the zero temperature behavior we will now discuss the case when we switch on a non-zero temperature i.e make $g(u) < 1$ or equivalently the inverse horizon radius, u_h finite in (3.128), where

$$g(u) = 1 - \frac{u^4}{u_h^4} \quad (3.168)$$

Choosing the same quark world line (3.127) and the string embedding (3.130) with the same boundary condition (3.131) but now in Euclidean space with compact time direction, the string action at finite temperature can be written as

$$S_{\text{NG}} = \frac{\tilde{T}}{2\pi} \int_{-\frac{d}{2}}^{+\frac{d}{2}} \frac{dx}{u^2} \sqrt{g(u) (\mathcal{A}_n u^n)^2 + \left[\mathcal{G}_m u^m - \frac{2g_s^2 \tilde{\mathcal{D}}_{n+m_o} \tilde{\mathcal{D}}_{l+m_o} \mathcal{A}_k u^{4+n+l+k+2m_o}}{u_h^4} \right] \left(\frac{\partial u}{\partial x} \right)^2} \quad (3.169)$$

where $\mathcal{G}_m u^m$ is defined in (3.139) and the correction to $\mathcal{G}_m u^m$ is suppressed by g_s^2 as well as u^4/u_h^4 because the background dilaton and non-zero temperature induces a slightly different world-sheet metric than what one would have naively taken. To avoid clutter, we will further redefine these corrections as:

$$\mathcal{G}_m u^m - \frac{2g_s^2 \tilde{\mathcal{D}}_{n+m_o} \tilde{\mathcal{D}}_{l+m_o} \mathcal{A}_k u^{4+n+l+k+2m_o}}{u_h^4} \equiv \tilde{\mathcal{D}}_l u^l \quad (3.170)$$

Minimizing this action gives the equation of motion for $u(x)$ and using the exact same procedure as for zero temperature, the corresponding equation for the distance between the quarks can be written as:

$$d = 2u_{\max} \int_0^1 dv \left\{ v^2 \sqrt{\tilde{\mathcal{D}}_m u_{\max}^m v^m} \frac{\sqrt{1 - \frac{u_{\max}^4}{u_h^4} \mathcal{A}_n u_{\max}^n}}{\left(1 - \frac{v^4 u_{\max}^4}{u_h^4}\right) (\mathcal{A}_m u_{\max}^m v^m)^2} \left[1 - v^4 \frac{\left(1 - \frac{u_{\max}^4}{u_h^4}\right)}{\left(1 - \frac{v^4 u_{\max}^4}{u_h^4}\right)} \left(\frac{\mathcal{A}_n u_{\max}^n}{\mathcal{A}_m u_{\max}^m v^m}\right)^2 \right]^{-1/2} \right\} \quad (3.171)$$

Once we have d , the renormalized Nambu-Goto action can also be written following similar procedure. The result is

$$S_{\text{NG}}^{\text{ren}} = \frac{\tilde{T}}{\pi} \frac{1}{u_{\max}} \left\{ -\hat{\mathcal{D}}_0 + \sum_{l=2} \frac{\hat{\mathcal{D}}_l}{l-1} - \int_0^1 \frac{dv}{v^2} \sqrt{\tilde{\mathcal{D}}_m u_{\max}^m v^m} + \mathcal{O}(g_s^2) \right. \\ \left. + \int_0^1 \frac{dv}{v^2} \sqrt{\tilde{\mathcal{D}}_m u_{\max}^m v^m} \left[1 - v^4 \frac{\left(1 - \frac{u_{\max}^4}{u_h^4}\right)}{\left(1 - \frac{v^4 u_{\max}^4}{u_h^4}\right)} \left(\frac{\mathcal{A}_n u_{\max}^n}{\mathcal{A}_m u_{\max}^m v^m}\right)^2 \right]^{-1/2} + \mathcal{O}(\epsilon_o) \right\} \quad (3.172)$$

which is somewhat similar in form with (3.153), which we reproduce in the limit $u_h \rightarrow \infty$. Also as in (3.153), we have defined $\sqrt{\tilde{\mathcal{D}}_m u_{\max}^m v^m} \equiv \hat{\mathcal{D}}_l v^l$.

Now just like the zero temperature case, requiring that d be real, sets an upper bound to u_{\max} , that we denote again by \mathbf{u}_{\max} , and is found by solving the following equation:

$$\frac{1}{2}(m+1)\mathcal{A}_{m+3}\mathbf{u}_{\max}^{m+3} + \frac{1}{j!} \prod_{k=0}^{j-1} \left(k - \frac{1}{2}\right) \left(\frac{\mathbf{u}_{\max}^4}{u_h^4}\right)^j \left[\mathcal{A}_l \mathbf{u}_{\max}^l \left(\frac{l}{2} + 2j - 1\right) \right] = 1 \quad (3.173)$$

Once we fix u_h and the coefficients of the warp factor \mathcal{A}_n , \mathbf{u}_{\max} will be known. We will assume that \mathbf{u}_{\max} lies in region 3.

Rest of the analysis is very similar to the zero temperature case, although the final conclusions would be quite different. We proceed further by defining certain new variables in the following way:

$$\begin{aligned}
\tilde{\mathcal{A}}_l &= \sum_m \frac{\mathcal{A}_m}{u_h^{l-m}} \frac{1}{\left(\frac{l-m}{4}\right)!} \prod_{k=0}^{\frac{l-m}{4}-1} \left(k - \frac{1}{2}\right), & l-m \geq 4 \\
&= 0 & l-m < 4 \\
&= \mathcal{A}_l & l-m = 0
\end{aligned} \tag{3.174}$$

As before, we observe that for $u_{\max} \rightarrow \mathbf{u}_{\max}$, both the integrals (3.171),(3.172) are dominated by the behavior of the integrand near $v \sim 1$, where we can write

$$\begin{aligned}
d &= 2 \frac{\sqrt{\tilde{\mathcal{D}}_m \mathbf{u}_{\max}^m} \mathbf{u}_{\max}}{\sqrt{1 - \frac{\mathbf{u}_{\max}^4}{u_h^4}} \mathcal{A}_n \mathbf{u}_{\max}^n} \int_0^1 \frac{dv}{\sqrt{\tilde{\mathbf{A}}(1-v) + \tilde{\mathbf{B}}(1-v)^2}} \\
&= -2 \frac{\sqrt{\tilde{\mathcal{D}}_m \mathbf{u}_{\max}^m} \mathbf{u}_{\max}}{\sqrt{1 - \frac{\mathbf{u}_{\max}^4}{u_h^4}} \mathcal{A}_n \mathbf{u}_{\max}^n} \left[\frac{\log \tilde{\mathbf{A}} - \log \left(2\sqrt{\tilde{\mathbf{B}}(\tilde{\mathbf{A}} + \tilde{\mathbf{B}})} + 2\tilde{\mathbf{B}} + \tilde{\mathbf{A}} \right)}{\sqrt{\tilde{\mathbf{B}}}} \right]
\end{aligned} \tag{3.175}$$

where taking the lower limit of the integral to 0 again do not change any of our conclusion. On the other hand, the renormalized Nambu-Goto action for the string now becomes:

$$\begin{aligned}
S_{\text{NG}}^{\text{ren}} &= \frac{\tilde{T}}{\pi} \frac{\sqrt{\tilde{\mathcal{D}}_m \mathbf{u}_{\max}^m}}{\mathbf{u}_{\max}} \left[\int_0^1 \frac{dv}{\sqrt{\tilde{\mathbf{A}}(1-v) + \tilde{\mathbf{B}}(1-v)^2}} - 1 \right] - \frac{\tilde{T}}{\pi \mathbf{u}_{\max}} + \mathcal{O}(\mathbf{u}_{\max}^2) \\
&= -\frac{\tilde{T}}{\pi} \frac{\sqrt{\tilde{\mathcal{D}}_m \mathbf{u}_{\max}^m}}{\mathbf{u}_{\max}} \left[\frac{\log \tilde{\mathbf{A}} - \log \left(2\sqrt{\tilde{\mathbf{B}}(\tilde{\mathbf{A}} + \tilde{\mathbf{B}})} + 2\tilde{\mathbf{B}} + \tilde{\mathbf{A}} \right)}{\sqrt{\tilde{\mathbf{B}}}} - 1 \right] \\
&\quad - \frac{\tilde{T}}{\pi \mathbf{u}_{\max}} + \mathcal{O}(\mathbf{u}_{\max}^2)
\end{aligned} \tag{3.176}$$

where $\tilde{\mathbf{A}}$ and $\tilde{\mathbf{B}}$ are defined exactly as in (3.162) but with \mathcal{A}_n replaced by $\tilde{\mathcal{A}}_n$ given by (3.174) above. It is also clear that:

$$\lim_{u_{\max} \rightarrow \mathbf{u}_{\max}} \tilde{\mathbf{A}} \rightarrow 0 \tag{3.177}$$

and so both (3.175) as well as (3.176) have identical logarithmic divergences. This would imply that the finite quantity is the ratio between (3.176) and (3.175):

$$\frac{S_{\text{NG}}^{\text{ren}}}{d} = \frac{\tilde{T}}{\pi} \left(1 - \frac{\mathbf{u}_{\max}^4}{u_h^4} \right)^{\frac{1}{2}} \frac{\mathcal{A}_n \mathbf{u}_{\max}^n}{\mathbf{u}_{\max}^2} \tag{3.178}$$

Now using the identity (3.126) and the above relation (3.166) we get our final result:

$$V_{Q\bar{Q}} = \sqrt{1 - \frac{\mathbf{u}_{\max}^4}{u_h^4}} \left(\frac{\mathcal{A}_n \mathbf{u}_{\max}^n}{\pi \mathbf{u}_{\max}^2} \right) d/\alpha' \quad (3.179)$$

This gives linear potential at large distances and from the form of the potential above, we can see that the higher the temperature, i.e. lower the u_h , the lower the slope of the linear term. In fact we will explicitly plot the potential at various temperature for certain choice of warp factors in section 3.3.4 and analyze this melting of the potential.

3.3.3 Linear Confinement from generic dual geometries

Having argued for linear confinement for geometries which are like region 3 of Fig 2.17, we will now consider a more general warp factor choice of h in dual geometry with metric (2.91). Much of the discussion here closely follows our recent work [186] and further details can be found there. We will restrict to cascading gauge theories where the effective number of colors grows as scale grows. This property of a gauge theory is most relevant for physical theories as new degrees of freedom emerge at UV and effective degrees of freedom shrink in IR to form condensates at low energy [46]. The number of colors at any scale $u = 1/r$ is given by (3.33) and for the analysis given here, it is simpler to define

$$\mathcal{H}(u) \equiv \frac{u^2}{\sqrt{h}} = \frac{\sqrt{N}}{L^2 \sqrt{N_{\text{eff}}}} \quad (3.180)$$

instead of $N_{\text{eff}}(u)$. The coefficients \mathcal{A}_n in the previous section are related to $\mathcal{H}(u)$ by $\mathcal{H}(u) = \mathcal{A}_n u^n$; and h is the warp factor. In terms of $\mathcal{H}(u)$, the condition that $N_{\text{eff}}(u)$ is a decreasing function of $u = 1/r$ becomes

$$\mathcal{H}'(u) > 0 \quad (3.181)$$

Combining Eqs.(3.180) and (3.181) yields the following condition

$$\mathcal{H}(u) > \frac{1}{L^2} \quad (3.182)$$

From now on, the value of L is set to 1 for the rest of this subsection, so that $\mathcal{H}(u) > 1$.

Zero temperature

Let u_{\max} be the maximum value of u for the string between the quark and the anti-quark. Then the relationship between u_{\max} and the distance between the quark and the anti-quark is given by [62]

$$d(u_{\max}) = 2u_{\max} \mathcal{H}(u_{\max}) \int_{\epsilon_0}^1 dv \frac{v^2 \sqrt{\mathcal{G}_m u_{\max}^m v^m}}{(\mathcal{H}(u_{\max} v))^2} \left[1 - v^4 \left(\frac{\mathcal{H}(u_{\max})}{\mathcal{H}(u_{\max} v)} \right)^2 \right]^{-1/2} \quad (3.183)$$

For the above expression to represent the physical distance between a quark and an anti-quark in vacuum, the integral must be real. This is guaranteed if for all $0 \leq v \leq 1$:

$$W(v|u_{\max}) \equiv v^2 \left(\frac{\mathcal{H}(u_{\max})}{\mathcal{H}(u_{\max} v)} \right) \leq 1 \quad (3.184)$$

For AdS space, (3.184) is automatic, as $\mathcal{H} = 1$ and then d is proportional to u_{\max} which results in only Coulomb potential. But for a generic warp factor, (3.184) gives rise to an upper bound for u_{\max} as already discussed.

To show confinement at large distances the potential between the quark and the anti-quark must be long ranged. That is, $d(u_{\max})$ must range from 0 to ∞ as u_{\max} varies from 0 to its upper bound, say $u_{\max} = x_{\max}$. Since $\mathcal{H}(u) > 1$, the only way to satisfy these conditions is via sufficiently fast vanishing of the square-root in Eq.(3.183) as $v \rightarrow 1$ at $u_{\max} = x_{\max}$.

For most u_{\max} , $1 - W(v|u_{\max})^2$ vanishes only linearly as v approaches 1. In this case, $d(u_{\max})$ is finite as the singularity in the integrand behaves like $1/\sqrt{1-v}$ and hence it is integrable. To make $d(u_{\max})$ diverge at $u_{\max} = x_{\max}$, $1 - W(v|x_{\max})^2$ must vanish quadratically as v approaches 1 to make the integrand sufficiently singular, $1/\sqrt{1 - W(v|x_{\max})^2} \sim 1/|1 - v|$. Therefore, the function $W(v|x_{\max})$ must have a maximum at $v = 1$.

To determine the value of x_{\max} , consider

$$W'(v|x_{\max}) = 2v \left(\frac{\mathcal{H}(x_{\max})}{\mathcal{H}(x_{\max} v)} \right) \left(1 - (x_{\max} v) \frac{\mathcal{H}'(x_{\max} v)}{2\mathcal{H}(x_{\max} v)} \right) \quad (3.185)$$

For this to vanish at $v = 1$, x_{\max} must be the smallest positive solution of

$$x\mathcal{H}'(x) - 2\mathcal{H}(x) = 0 \quad (3.186)$$

With the definition $\mathcal{H}(u) = \mathcal{A}_n u^n$, one can easily show that this is equivalent to the condition (3.159) which was originally derived in [62]. The allowed range of u_{\max} is then

$$0 \leq u_{\max} \leq x_{\max} \quad (3.187)$$

and within this range, $d(u_{\max})$ varies from 0 to ∞ . How it varies will depend on the values of \mathcal{G}_m as well as $\mathcal{H}(u)$.

Finite temperature

At finite temperature, the relation between u_{\max} and the distance between the quark and the anti-quark is obtained by replacing $\mathcal{H}(u)$ with $\sqrt{1 - u^4/u_h^4} \mathcal{H}(u)$ in Eq.(3.183):

$$\begin{aligned} d_T(u_{\max}) &= 2u_{\max} \sqrt{1 - u_{\max}^4/u_h^4} \mathcal{H}(u_{\max}) \int_{\epsilon_0}^1 dv \frac{v^2 \sqrt{\mathcal{D}_m u_{\max}^m v^m}}{(1 - v^4 u_{\max}^4/u_h^4) (\mathcal{H}(u_{\max} v))^2} \\ &\times \left[1 - v^4 \frac{(1 - u_{\max}^4/u_h^4)}{(1 - v^4 u_{\max}^4/u_h^4)} \left(\frac{\mathcal{H}(u_{\max})}{\mathcal{H}(u_{\max} v)} \right)^2 \right]^{-1/2} \end{aligned} \quad (3.188)$$

The explicit factor of u_{\max} makes $d_T(u_{\max})$ vanish at $u_{\max} = 0$ as in the $T = 0$ case.

As u_{\max} approaches u_h , the integral near $v = 1$ behaves like

$$d_T(u_{\max}) \sim \int_0^1 dv \frac{\sqrt{1 - u_{\max}^4/u_h^4}}{\sqrt{(1 - v)(1 - v u_{\max}/u_h)}} \quad (3.189)$$

which indicates that $d_T(u_{\max})$ goes to 0 as u_{\max} approaches u_h . Hence, at both $u_{\max} = 0$ and $u_{\max} = u_h$, $d_T(u_{\max})$ vanishes. Since $d_T(u_{\max})$ is positive in general, there has to be a maximum between $u_{\max} = 0$ and $u_{\max} = u_h$. Whether the maximum value of $d_T(u_{\max})$ is infinite as in the $T = 0$ case depends on the temperature (equivalently, u_h^{-1}) as we now show.

The fact that the physical distance needs to be real yields the following condition.

For all $0 \leq v \leq 1$,

$$W_T(v|u_{\max}) \equiv v^2 \left(\frac{\mathcal{H}(u_{\max})}{\mathcal{H}(u_{\max} v)} \right) \sqrt{\frac{1 - u_{\max}^4/u_h^4}{1 - u_{\max}^4 v^4/u_h^4}} \leq 1 \quad (3.190)$$

Taking the derivative gives

$$\begin{aligned}
W'_T(v|u_{\max}) &= \frac{(1 - u_{\max}/u_h^4)^{1/2}}{(1 - u_{\max}^4 v^4/u_h^4)^{3/2}} \frac{v\mathcal{H}(u_{\max})}{\mathcal{H}(u_{\max}v)} \\
&\times \left[-(u_{\max}v)(1 - (u_{\max}v/u_h)^4)\mathcal{H}'(u_{\max}v) + 2\mathcal{H}(u_{\max}v) \right]
\end{aligned} \tag{3.191}$$

Similarly to the $T = 0$ case, $d_T(u_{\max})$ can have an infinite range if the derivative vanishes at $v = 1$ for a certain value of u_{\max} , say $u_{\max} = y_{\max}$. This value of y_{\max} is determined by the smallest positive solution of the following equation

$$y\mathcal{H}'(y) - 2\mathcal{H}(y) = (y/u_h)^4 y\mathcal{H}'(y) \tag{3.192}$$

which then forces $W'_T(1|y_{\max})$ to vanish. Note that the left hand side is the same as the zero temperature condition, Eq.(3.186). The right hand side is the temperature (u_h) dependent part. Using the facts that:

$$\prod_{k=0}^{j-1} (k - 1/2) = -(2j - 3)!!/2^j, \quad \sum_{j=1}^{\infty} x^j (2j - 3)!!/2^j j! = 1 - \sqrt{1 - x}, \tag{3.193}$$

it can be readily shown that Eq.(3.192) is equivalent to Eq.(3.173) as long as $u_{\max} < u_h$. It is also clear that $y = u_h$ cannot be a solution of Eq.(3.192) because at $y = u_h$, the equation reduces to $\mathcal{H}(u_h) = 0$ which is inconsistent with the fact that $\mathcal{H}(y) \geq 1$.

Recall that we are considering gauge theories for which $\mathcal{H}(y) \geq 1$ and $\mathcal{H}'(y) \geq 0$, and we assume that the equation $y\mathcal{H}'(y) - 2\mathcal{H}(y) = 0$ has a real positive solution x_{\max} which gives confinement at zero temperature. Hence as y increases from 0 towards x_{\max} , the left hand side of Eq.(3.192) increases from -2 while the right hand side increases from 0. The left hand side reaches 0 when $y = x_{\max}$ which is the point where the distance $d(u_{\max})$ at $T = 0$ becomes infinite. At this point the right hand side of Eq.(3.192) is positive and has the value $(x_{\max}/u_h)^4 x_{\max} \mathcal{H}'(x_{\max})$. Hence the solution of Eq.(3.192), if it exists, must be larger than x_{\max} .

Consider first low enough temperatures so that $u_h \gg x_{\max}$. For these low temperatures, Eq.(3.192) will have a solution, as the right hand side will be still small around $y = x_{\max}$. This then implies that the linear potential at low temperature will have an infinite range if the zero temperature potential has an infinite range.

Now we show that the infinite range potential cannot be maintained at all temperatures. We can have a black hole such that $u_h = x_{\max}$. When the left hand side vanishes at $y = x_{\max}$, the right hand side is $x_{\max} \mathcal{H}'(x_{\max}) = 2\mathcal{H}(x_{\max})$ which is positive and finite. For $y > x_{\max}$, the left hand side $(y\mathcal{H}'(y) - 2\mathcal{H}(y))$ may become positive, but it is always smaller than $y\mathcal{H}'(y)$ since $\mathcal{H}(y)$ is always positive. But for the same y , the right hand side $((y/u_h)^4 y\mathcal{H}'(y))$ is always positive and necessarily larger than $y\mathcal{H}'(y)$ since $(y/u_h) > 1$. Hence, Eq.(3.192) cannot have a real and positive solution when $u_h = x_{\max}$. Therefore between $u_h = \infty$ and $u_h = x_{\max}$, there must be a point when Eq.(3.192) cease to have a positive solution.

When Eq.(3.192) has no solution, then the expression for $d_T(u_{\max})$, (3.188) will not diverge for any u_{\max} within $(0, u_h)$. Furthermore, since the expression vanishes at both ends, there must be a maximum $d_T(u_{\max})$ at a non-zero u_{\max} . When the distance between the quark and the anti-quark is greater than this maximum distance, there can no longer be a string connecting the quark and the anti-quark.

To summarize, we have just shown that if we start with a dual geometry that allows infinite range linear potential at zero temperature, there exists some critical temperature above which the string connecting the quarks breaks. This shows that at high enough temperatures quarkonium state melts and gives rise to 'free quarks'. In the following subsection, we will try to quantify the melting temperature using geometries with exponential warp factors.

3.3.4 Numerical analysis of melting temperatures

After discussing the most general choice for warp factors that give rise to y_{\max} and consequently linear potential, we will now give specific examples of geometries that may arise as solutions to Einstein's equation. We start with the following ansatz for the metric:

$$\begin{aligned} ds^2 &= -\frac{g}{\sqrt{h}} dt^2 + \frac{1}{\sqrt{h}} (dx^2 + dy^2 + dz^2) + \frac{\sqrt{h}}{u^2} \left(\frac{H}{gu^2} du^2 + ds_{\mathcal{M}_5}^2 \right) \\ &\equiv -\frac{g}{\sqrt{h}} dt^2 + \frac{1}{\sqrt{h}} (dx^2 + dy^2 + dz^2) + \frac{\sqrt{h}}{u^2} \tilde{g}_{mn} dx^m dx^n \end{aligned} \quad (3.194)$$

where $h \equiv h(u, \theta_i, \phi_i, \psi)$, $H \equiv H(u, \theta_i, \phi_i, \psi)$, and $g \equiv 1 - u^4/u_h^4$; \mathcal{M}_5 is the compact five dimensional manifold parametrized by coordinates (θ_i, ϕ_i, ψ) and can be thought of as a perturbation over $T^{1,1}$. Here $u = 0$ is the boundary and $u = u_h$ is the horizon. As discussed in [62], the above metric arises in region 3 of [62] when one considers the running of axio-dilaton τ , $D7$ brane local action and fluxes due to anti five-branes on a geometry that deviates from the IR OKS-BH geometry from the back reactions of the above sources. The three-form fluxes sourced by (p, q) anti-branes are proportional to $r^{-i}f(r)$ for some positive i (see [62] for details about $f(r)$), where the function $f(r) \rightarrow 1$ as $r \rightarrow \infty$ and $f(r) \rightarrow 0$ as $r \rightarrow 0$. With the coordinate $u = 1/r$, there is another function: $k(u) \equiv \exp(-u^A)$, $A > 0$, that also has somewhat similar behavior as $f(u)$ and may allow us to have a better analytic control on the background. With such a choice of $k(u)$, the total three form flux is proportional to $u^A M(u)$ with

$$M(u) \equiv M[1 - k(u)] = M[1 - \exp(-u^A)] \quad (3.195)$$

where M is the number of bi-fundamental flavors. Thus three-form fluxes are decaying fast as $Mu^A[1 - \exp(-u^A)]$ and, as shown in [62], the seven-branes could be arranged such that the axio-dilaton τ behaves typically as $\tau \sim u^B$. This means that from the behavior of the internal Riemann tensor one may conclude that the internal metric \tilde{g}_{mn} behaves as $\tilde{g}_{mn} \sim u^C \exp(c_o u^C)$ where A, C, A and C are all positive and c_o could be positive or negative depending on the precise background informations.

From the above discussions it should be clear that taking the three-forms and world-volume gauge fluxes to be exponentially decaying in the IR (but axio-dilaton to be suppressed only as u^B) should solve all the equations of motion, giving the following behavior for the warp factor h and the internal metric H in (3.194)²⁰:

$$h = L^4 u^4 \exp(-\alpha u^{\tilde{\alpha}}), \quad H = \exp(\beta u^{\tilde{\beta}}) \quad (3.196)$$

where we are taking $\alpha, \tilde{\alpha}, \beta, \tilde{\beta}$ to be all positives with α, β to be functions of internal coordinates (θ_i, ϕ_i, ψ) and $L^4 = g_s N \alpha'^2$ to be the asymptotic AdS throat radius²¹.

²⁰See also the interesting works of [187] where exponential warp factors have been chosen.

²¹Note that β in (3.196) could be considered negative so that H would be decaying to zero in

Motivated by the above arguments, we will consider Nambu-Goto action of the string in the geometry with $(\tilde{\alpha}, \tilde{\beta}) = (3, 3)$ and $(\tilde{\alpha}, \tilde{\beta}) = (4, 4)$ at temperatures $T^{(1)}$ and $T^{(2)}$ respectively in Eq.(3.196). As in [57][62] we consider mappings $X^\mu(\sigma, \tau)$, which are points in the internal space, to lie on the slice:

$$\theta_1 = \theta_2 = \pi, \quad \phi_i = 0, \quad \psi = 0 \quad (3.197)$$

so that on this slice α, β are fixed and we set it to $(\alpha, \beta) = (0.1, 0.05)$ for both choices $(\tilde{\alpha}, \tilde{\beta})$. (Such a choice of slice will also help us to ignore the three-form contributions to the Wilson loop.) With these fixed choices for the warp factors, we plot the inter quark separation d as a function of u_{\max} in Figures 3.3 and 3.4 for various values of $T \equiv 1/u_h$.

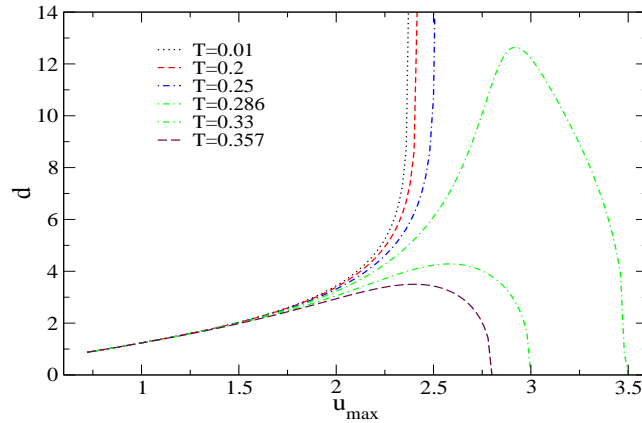


Figure 3.3: Inter quark distance as a function of u_{\max} for various temperatures and warp factor with $(\alpha, \tilde{\alpha}, \beta, \tilde{\beta}) = (0.1, 3, 0.05, 3)$ in the warp factor equation.

Note that for both choices of warp factors, for low enough temperatures, there exist $u_{\max} = y_{\max}$ where $d \rightarrow \infty$. As the temperature is increased, y_{\max} increases modestly. On the other hand from figure 3.3, one sees that when $T > T_c^{(1)} \sim 0.28$ there exists a d_{\max} which is finite. This means for inter quark distance $d > d_{\max}$, there

the IR. However since region 3 doesn't extend to the IR we don't have to worry about the far IR behavior of Eq.(3.196).

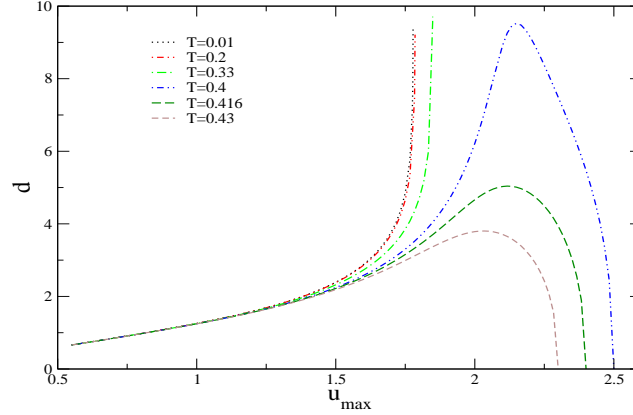


Figure 3.4: Quark-anti quark distance as a function of u_{\max} for various temperatures and warp factor with $(\alpha, \tilde{\alpha}, \beta, \tilde{\beta}) = (0.1, 4, 0.05, 4)$ in the warp factor equation.

is *no* string configuration with boundary condition $x(0) = \pm d/2$ implying that the string attaching the quarks breaks and we have two *free* partons for $d > d_{\max}$. Thus we can interpret d_{\max} to be a “screening length”. From Fig 3.4 we observe similar behavior but now d_{\max} exists for $T > T_c^{(2)} \sim 0.399$.

In figure 3.5, d_{\max} as a function of T is plotted. We note that for a small change in the temperature near $T_c^{(1)}$ (or near $T_c^{(2)}$ equivalently) there is a sharp decrease in screening length d_{\max} , but for $T \gg T_c^{(i)}$, $i = 1, 2$, the screening length does not change much. In fact d_{\max} behaves as $C + \exp(-\gamma T)$ (where C and γ are constants) which in turn could be an indicative of a phase transition near $T_c^{(i)}$ for $i = 1, 2$ i.e the two choices of warp factor.

Finally we plot the potential energy $V_{Q\bar{Q}}$ as a function of d in Figures 3.6 and 3.7 for the two choices of warp factor. For $T < T_c^{(1)}$ in Fig 3.6 and $T < T_c^{(2)}$ in Fig 3.7, we have energies linearly increasing with an arbitrarily large increment of the inter quark separations. Thus we have linear confinement of quarks for large distances and small enough temperatures. For $T > T_c^{(i)}$, $i = 1$ or 2 , there exists a d_{\max} and for all distances $d > d_{\max}$ there are no Nambu-Goto actions, S_{NG} , for the string attaching *both* the quarks. This means that we have free quarks and $V_{Q\bar{Q}}$ is constant

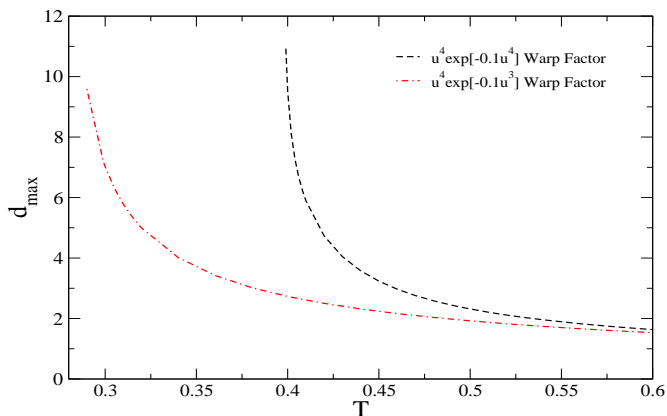


Figure 3.5: Maximum inter quark separation d_{\max} as a function of $T = 1/u_h$ for both cubic and quartic warp factors.

for $d > d_{\max}$. Of course looking at Fig 3.6 and 3.7 one shouldn't conclude that the free energy *stops* abruptly. What happens for those two cases is that the string joining the quarks breaks, and then the free energy is given by the sum of the energies of the two strings (from the tips of the seven-branes to the black-hole horizon) and the total energies of the small fluctuations on the world-volume of the strings. The latter contributions are non-trivial to compute and we will not address these in any more detail here, but energy conservation should tell us how to extrapolate the curves in Fig 3.6 and 3.7, beyond the points where the string breaks, for all $T > T_c$. Of course after sufficiently long time the two strings would dissipate their energies associated with their world-volume fluctuations and settle down to their lowest energy states.

To compare with lattice QCD calculation of free energy [188], in Fig 3.8, we plot side by side the lattice results and our calculation with $(\alpha, \tilde{\alpha}, \beta, \tilde{\beta}) = (0.1, 3, 0.05, 3)$ where the potential energy has been extrapolated to account for the energy of the disjoint strings i.e. the free quarks for $d > d_{\max}$. We observe the striking similarity between the shape of curves in the two plots. Two completely different approaches yield very similar results which only strengthens the validity of applying gauge/gravity correspondence in the study of strongly coupled QCD. Qualitatively the curves show

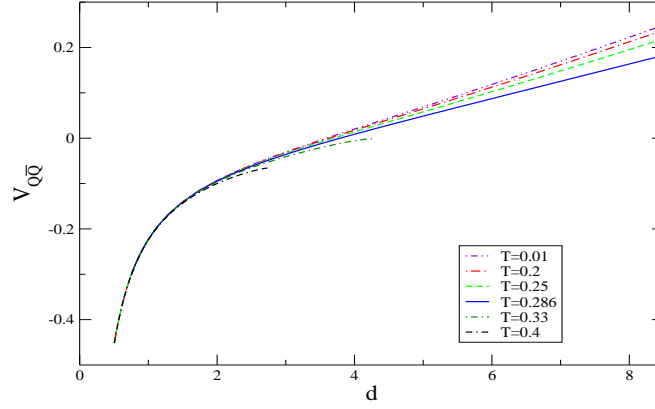


Figure 3.6: Heavy quark potential $V_{Q\bar{Q}}$ as a function of quark separation d with cubic warp factor, or equivalently, $(\alpha, \tilde{\alpha}, \beta, \tilde{\beta}) = (0.1, 3, 0.05, 3)$ in the warp factor equation for various temperatures.

how linear potential melts at high temperatures and there is also evidence of screening of the Coulomb potential at small separation of the quarks.

Observe that for a wide range of temperatures $0 < T < T_c^{(i)}$, the potential and thus the free energy hardly changes. But near a narrow range of temperatures $T_c^{(i)} - \epsilon < T < T_c^{(i)} + \epsilon$ (where $\epsilon \sim 0.05$), free energy changes significantly. For Fig 3.6 the change is more abrupt than Fig 3.7. This means as we go for bigger values of $\tilde{\alpha}$, the change in free energy is sharper.

In Fig 3.9, we plot the slope of the linear potential as a function of T . Again for a wide range of temperatures, there are no significant changes in the slope but near $T_c^{(i)}$, the change is more dramatic: the slope decreases sharply, indicating again the possibility of a phase transition near $T_c^{(i)}$. As we noticed before, here too bigger exponent $\tilde{\alpha}$ gives a sharper decline in the slope hinting that when $\tilde{\alpha} \gg 1$, the transition would be more manifest.

To conclude, the above numerical analysis suggest the presence of a deconfinement transition, where for a narrow range of temperatures $0.28 \leq T_c \leq 0.39$ the free energy of $Q\bar{Q}$ pair shows a sharp decline. Interestingly, changing the powers of u

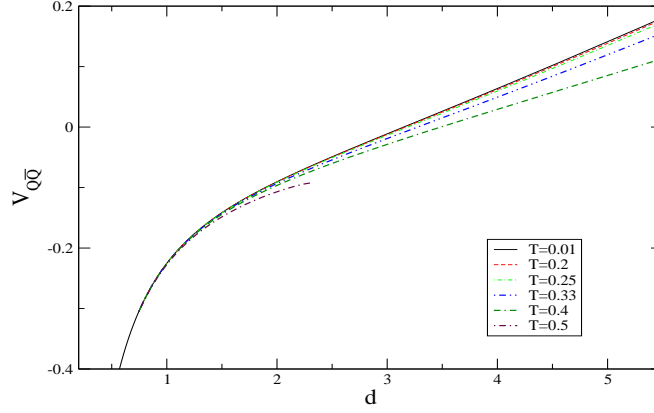


Figure 3.7: Heavy quark potential $V_{Q\bar{Q}}$ as a function of quark separation d with quartic warp factor, or equivalently, $(\alpha, \tilde{\alpha}, \beta, \tilde{\beta}) = (0.1, 4, 0.05, 4)$ in the warp factor equation for various temperatures.

in the exponential changes the range of T_c only by a small amount. So effectively T_c lies in the range $0.2 \leq T_c \leq 0.4$. Putting back units, and defining the *boundary* temperature²² \mathcal{T} as $\mathcal{T} \equiv \frac{g'(u_h)}{4\pi\sqrt{h(u_h)}}$, our analysis reveal:

$$\frac{0.91}{L^2} \leq \mathcal{T}_c \leq \frac{1.06}{L^2} \quad (3.198)$$

which is the range of the melting temperatures in these class of theories for heavy quarkonium states. Since the temperatures at both ends do not differ very much, this tells us that the melting temperature is inversely related to the asymptotic AdS radius in large N thermal QCD.

²²See sec. (3.1) of [57] for details.

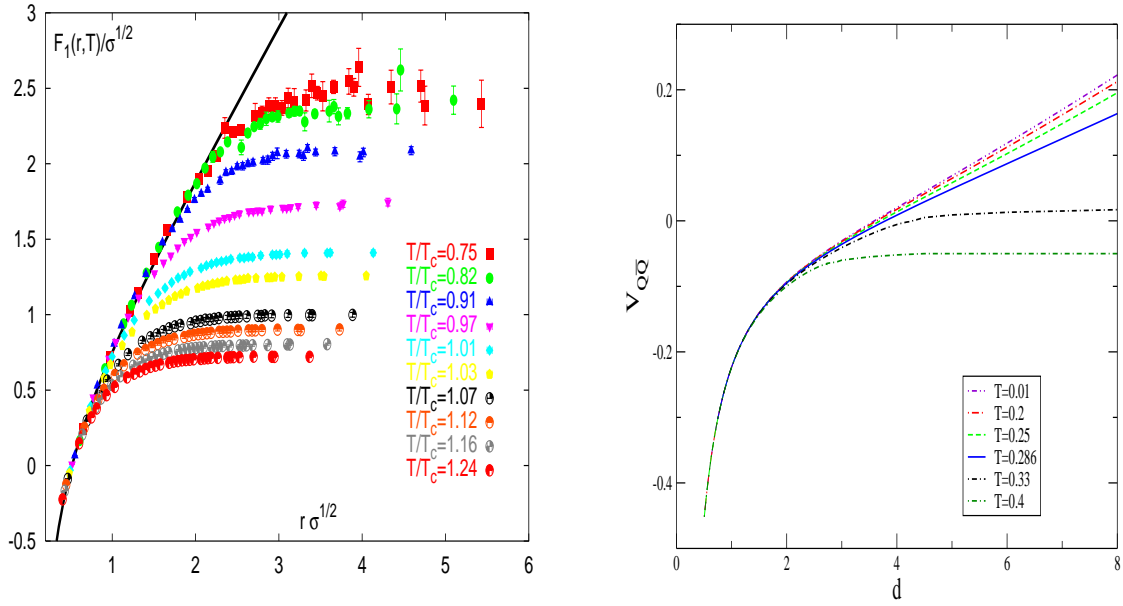


Figure 3.8: Comparison between lattice QCD results [188] and our analysis. The left figure is the lattice plot whereas the right figure is our calculation for the potential.

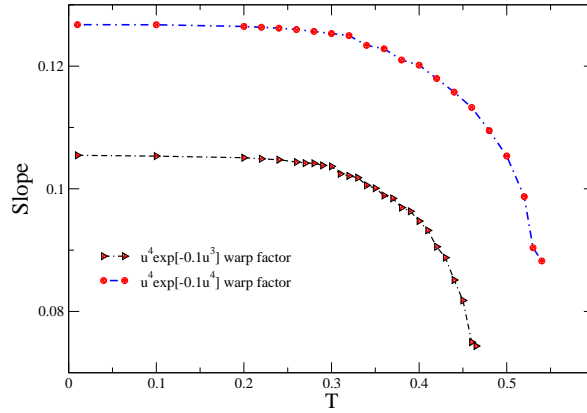


Figure 3.9: Slope of linear potential as a function of T for both cubic and quartic warp factors. Note that in the figure the slopes have been computed as $\frac{\Delta V}{\Delta d}$ with a range of d from 1.6 to 1.7 in appropriate units.

Chapter 4

Conclusions

In this thesis we have proposed the gravity dual for a non conformal finite temperature field theory with matter in fundamental representation. The gauge couplings run logarithmically in the IR while in the UV they become almost constant and the theory approaches conformal fixed point. To our knowledge, the brane construction and the dual geometry in section 2.4 is the first attempt to UV complete a Klebanov-Strassler type gauge theory with asymptotically conformal field theory.

Although our construction is rather technical with the gauge group being of the form $SU(N + M) \times SU(N)$ in the IR, one can perform a cascade of Seiberg dualities to obtain the group $SU(\bar{M})$ and identify this with strongly coupled QCD. One may even interpret that the gauge groups depicted in Fig 2.19 contain strongly coupled large N QCD. This is indeed consistent as the coupling g_{N+M} of $SU(N + M)$ factor in $SU(N + M) \times SU(N + M)$ or in $SU(N + M) \times SU(N)$, *always* decreases as scale is increased. To see this, observe that in the IR, the coupling g_{N+M} runs logarithmically with scale and gets stronger as scale is decreased. At the UV, we can arrange the sources in the dual geometry such that g_{N+M} decreases as the scale grows and runs as $g_{M+N} \sim a_k/\Lambda^k$. By demanding $-ka_k/\Lambda^{k-1} < 0$, we see that g_{N+M} indeed decreases as scale is increased. Thus from UV to IR coupling always increases, just like QCD.

Of course the 't Hooft coupling $\lambda_{N+M} = (N + M)g_{N+M}$ for the group $SU(N + M)$ is still large in the limit $N \rightarrow \infty$, even if g_{N+M} has decreased to very small value. This allows us to use the classical dual gravity description for the gauge theory which

has a coupling that shrinks in the UV, mimicking QCD. For even higher energies, 't Hooft coupling will eventually become too small for finite $N + M$ and supergravity description will no longer hold. But we can use perturbative methods to analyze the field theory in the highest energies. For finite $N + M$ and very high energies, the gauge coupling may even vanish giving rise to asymptotically free theory. These arguments lead us to conclude that, in principle, the brane configuration we proposed can incorporate QCD and in the large N limit, the gravity dual we constructed can capture features of QCD.

Using the dual geometry, we have studied the dynamics of 'quarks' in the gauge theory and computed shear viscosity η and its ratio to entropy η/s . The key to most of our analysis was the calculation of the stress tensor of gauge theory and we showed how different UV completions contribute to its expectation value. Using the correlation function for the stress tensor, we computed the shear viscosity of the medium while the computation of pressure and energy density allowed us to calculate the entropy of the system. Using a similar procedure to calculate correlators of stress energy tensors, with introducing diagonal perturbations in the background metric, we can easily evaluate the bulk viscosity ζ of the non-conformal fluid. One can consider vector and tensor fields of higher rank to couple to the graviton perturbations in the five dimensional effective theory and study how this coupling affects the bulk viscosity. The calculation is underway and we hope to report on it in the near future.

All the calculations we performed regarding properties of the plasma did not account the effect of expansion of the medium which is crucial in analyzing fluid dynamics. One possible improvement would be to construct a time dependent dual gravity which can describe the expansion of the QGP formed in heavy ion collisions. A first attempt would be to consider collisions of open strings ending on D7 branes and then compute their back-reactions on the geometry. The gravity waves associated with the collisions evolve with time and from the induced boundary metric, one can compute the energy momentum tensor of the field theory. Analyzing the time dependence of this stress tensor, one can learn about the evolution of the medium and subsequently account for the effects it has on the quark dynamics.

We have computed the fluxes and the form of the warp factor for our static dual geometry, but did not give explicit expressions for the deformation of the internal five dimensional metric to all orders in $g_s N_f$. However, we have explicitly shown the Einstein equations that determine the form of the internal metric and using our ansatz in [62], one can in principle compute coefficients in the expansion of the internal metric to all orders in $g_s N_f$. Even without a precise knowledge of the internal geometry, we were able to extract crucial information about the dual gauge theory and formulated how the higher order corrections may enter into our analysis. Most of the calculation only relied on the warp factor and with the knowledge of its precise form, we were able to calculate thermal mass, drag and diffusion coefficients, η/s and finally free energy of $Q\bar{Q}$ pair. For completeness of the supergravity analysis, we hope to compute the exact solution for the internal metric using the ansatz of [62] in our future work.

In our computation of the heavy quark potential, we classified the most general dual gravity that allows linear confinement of quarks at large separation and small temperatures. We showed that if a gauge theory has dual gravity description and its effective degrees of freedom grows monotonically in the UV, it always shows linear confinement at large distances, as long as the dual warp factor satisfies a very simple relation given by (3.192). Thus (3.192) can be regarded as a sufficient condition for linear confinement of gauge theories with dual gravity. It would be interesting to study what are the general brane configurations that allow warp factors which satisfy (3.192) and thus give rise to confining gauge theories. We leave it as a future direction to be explored.

As the dual geometry incorporates features of Seiberg duality cascade, our construction is ideal for studying phase transitions. The gauge theories we studied have description in terms of gauge groups of lower and lower rank. From the gravity dual analysis, by cutting the geometry at certain radial location and attaching another geometry up to infinity, we can construct gravity description for various phases of a gauge theory. Each phase will have different dual geometries attached in the large r region while the small r region will be common to all the theories. A flow from large r to small r geometry can be interpreted as ‘flow’ from an effective theory in

the UV to another in the IR. From UV to IR, the different effective theories will describe different phases of the gauge theory and this can allow one to study the various phases of dense matter. Thus our construction is not only useful to analyze strongly coupled gauge theory, but also has potential for studying phase transitions in ultra dense medium and we hope to address this issue in the future.

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